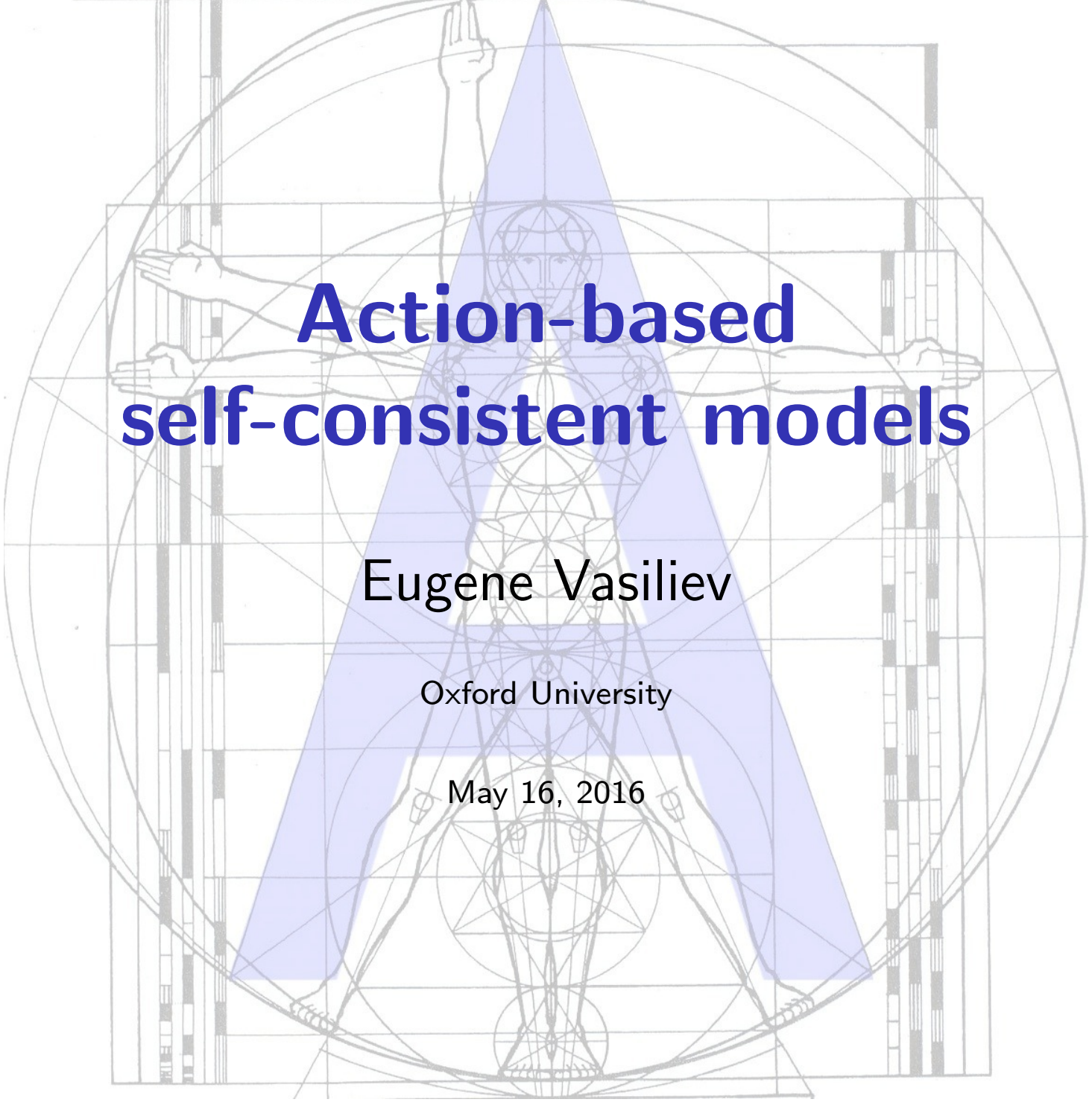

# Action-based self-consistent models

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May 16, 2016

# Plan of the talk

Overview of self-consistent modelling

Actions as the integrals of motion

Models specified by distribution functions

Non-parametric distribution functions

Conclusions

# Fundamental equations

1. Collisionless Boltzmann equation:

$$\frac{\partial f}{\partial t} + \mathbf{v}\,\frac{\partial f}{\partial \mathbf{x}} - \frac{\partial \Phi}{\partial \mathbf{x}}\,\frac{\partial f}{\partial \mathbf{v}} = 0.$$

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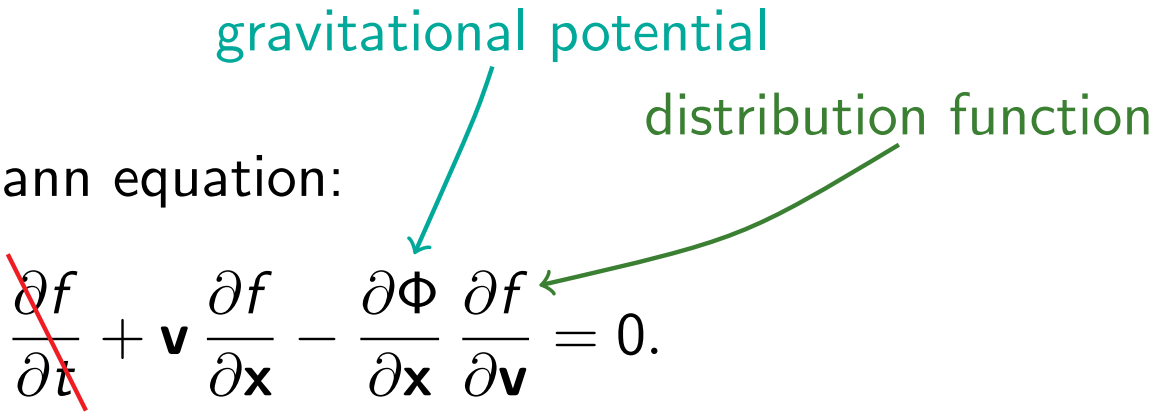


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(Assumption: a galaxy is a collisionless system in a steady state)

# Fundamental equations

gravitational potential

distribution function

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density

$$\nabla^2 \Phi(\mathbf{x}) = 4\pi\, G\, \rho(\mathbf{x}).$$

(Assumption: Newtonian gravity)

# Fundamental equations

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$$\nabla^2 \Phi(\mathbf{x}) = 4\pi\, G\, \rho(\mathbf{x}).$$

(Assumption: Newtonian gravity)

3. The link:

$$\rho(\mathbf{x}) = \iiint d^3 v\, f(\mathbf{x}, \mathbf{v}).$$

(Assumption: self-consistency)

# Ways of solving the equations

1. Collisionless Boltzmann equation:

   Jeans theorem states that the DF may depend only on the integrals of motion

$$f = f\big(\mathcal{I}(\mathbf{x},\mathbf{v})\big)\ , \quad \mathcal{I} = \{E, L, \dots\}.$$

   depend on the potential $\Phi$

2. Poisson equation:

$$\Phi(\mathbf{x}) = -\iiint d^3x'\, G\, \rho(\mathbf{x}') \times \frac{1}{|\mathbf{x}-\mathbf{x}'|}.$$

3. The link:

$$\rho(\mathbf{x}) = \iiint d^3v\, f(\mathbf{x},\mathbf{v}).$$

# Self-consistent models of spherical systems

- Integrals of motion:

  energy $E = \Phi(r) + v^2/2$, angular momentum $L = |\mathbf{x} \times \mathbf{v}|$.

- Gravitational potential:

$$\Phi(\mathbf{x}) \equiv \Phi(r) = -4\pi G \left[ r^{-1} \int_0^r dr'\, \rho(r')\, r'^{\,2} + \int_r^\infty dr'\, \rho(r')\, r'^{\,-1} \right].$$

- Distribution function:

  $f(E)$ (isotropic) or $f(E, L)$ (general anisotropic);
  in the isotropic case

$$\begin{aligned} \rho(r) &= \int_0^{v_{\rm esc}} 4\pi v^2\, dv\; f(\Phi(r) + v^2/2) \\ &= \int_{\Phi(r)}^0 dE\, 4\pi f(E) \sqrt{2[E - \Phi(r)]} \end{aligned}$$

## Two possible approaches

**1.** From $\rho$ to $f$: first compute $\Phi(r)$ and its inverse $r(\Phi)$;

$$\rho(r) = \int_{\Phi(r)}^{0} dE\, 4\pi f(E) \sqrt{2[E - \Phi(r)]} \quad \Longrightarrow$$

$$f(E) = \frac{d}{dE} \int_{E}^{0} d\Phi\, \frac{d\rho\big(r(\Phi)\big)}{d\Phi}\, \frac{1}{4\pi\sqrt{2[E - \Phi(r)]}}$$

(Eddington inversion), or its generalizations for anisotropic $f(E, L)$ (Ossipkov–Merritt, Cuddeford, etc.)

Examples: isochrone, Hernquist model, etc.

**2.** From $f$ to $\rho$:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Phi(r)}{dr}\right) = 4\pi G \iiint d^3v\, f\big(\Phi(r) + v^2/2,\ |\mathbf{x} \times \mathbf{v}|\big)$$

Examples: Plummer, King, lowered isothermal models, etc.

# Axisymmetric systems

- Integrals of motion:

  $E = \Phi(R,z) + v^2/2, \quad L_z = R\,v_\phi,$
  third integral $I_3 = ?$ (known analytically in special cases).

- Gravitational potential:

  $\Phi(R,z) = ?$ (can be reduced to a 1d integral over $\rho(R,z)$ in special cases, e.g., for $\rho(R^2 + z^2/q^2)$ or $\rho_R(R)\,\rho_z(z)$, etc.)

- Distribution function:

  $f(E, L_z)$ (two-integral) or $f(E, L_z, I_3)$ (general three-integral).

# Axisymmetric systems – two approaches

1. From $\rho(R, z)$ to $f$:
   - Compute $\Phi(R, z)$;
   - Express $z$ as a function of $R, \Phi$;
   - the two-integral $f(E, L_z)$ is obtained by a contour integral over $\rho(R, \Phi)$ [Hunter&Qian 1993].

2. From $f(E, L_z)$ to $\Phi$ and $\rho$:

   Iterative approach [e.g., Prendergast&Tomer 1970]:
   - Assume an initial guess for $\Phi(R, z)$;
   - Compute $\rho(R, z) = \iiint d^3v \, f\left(\Phi(R, z) + v^2/2, \; r\, v_\phi\right)$;
   - Find new $\Phi(R, z)$ from the Poisson equation;
   - Repeat until convergence.

Generalization to $f(E, L_z, I_3)$ – ? (straightforward if we know $I_3$...)

# Actions as integrals of motion

- One may use any set of integrals of motion, **but** actions are special:
- For bounded multiperiodic motion, actions are defined as $J = \frac{1}{2\pi} \oint \mathbf{p}\, d\mathbf{x}$, where $\mathbf{p}$ are canonically conjugate momenta for $\mathbf{x}$
- Action/angle variables $\{\mathbf{J}, \boldsymbol{\theta}\}$ are the most natural way of describing the motion: from Hamilton's equations we have

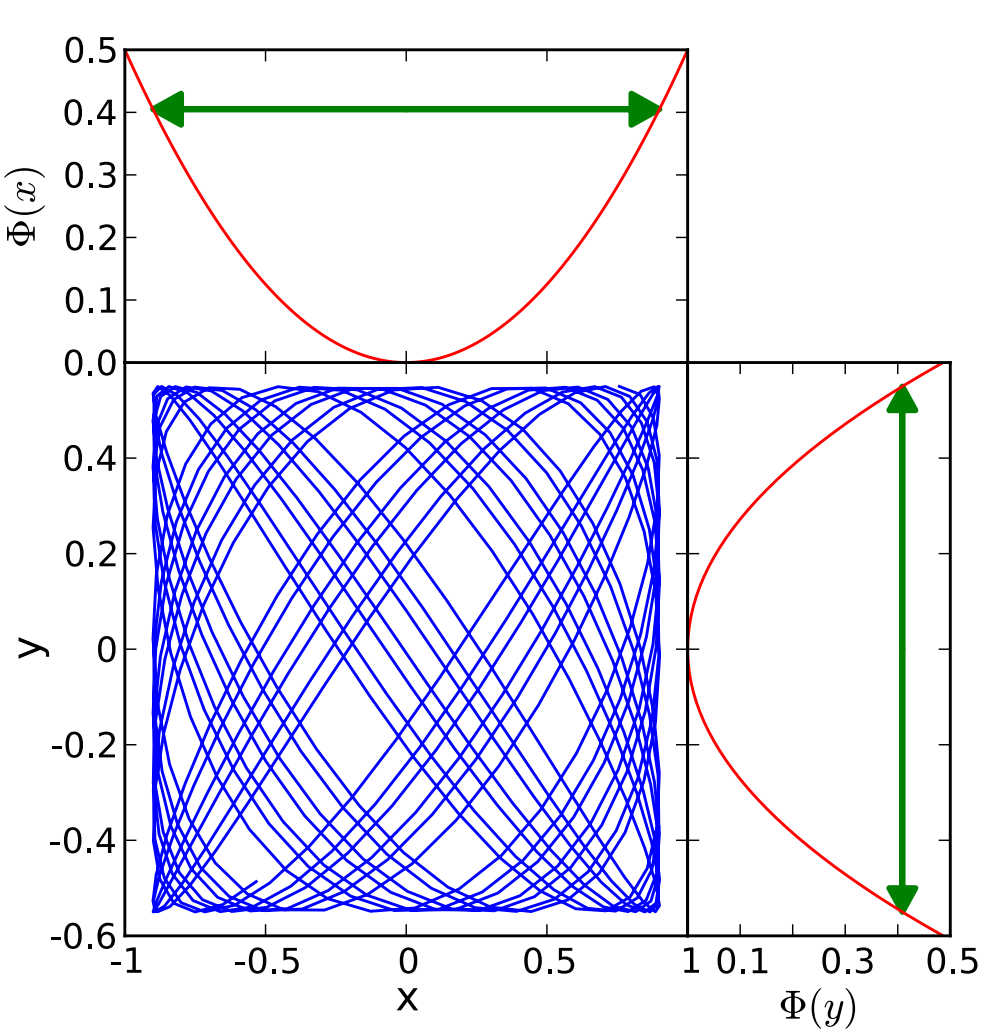
  $\frac{dJ_i}{dt} = -\frac{\partial H}{\partial \theta_i} = 0$ (actions are integrals of motion), and

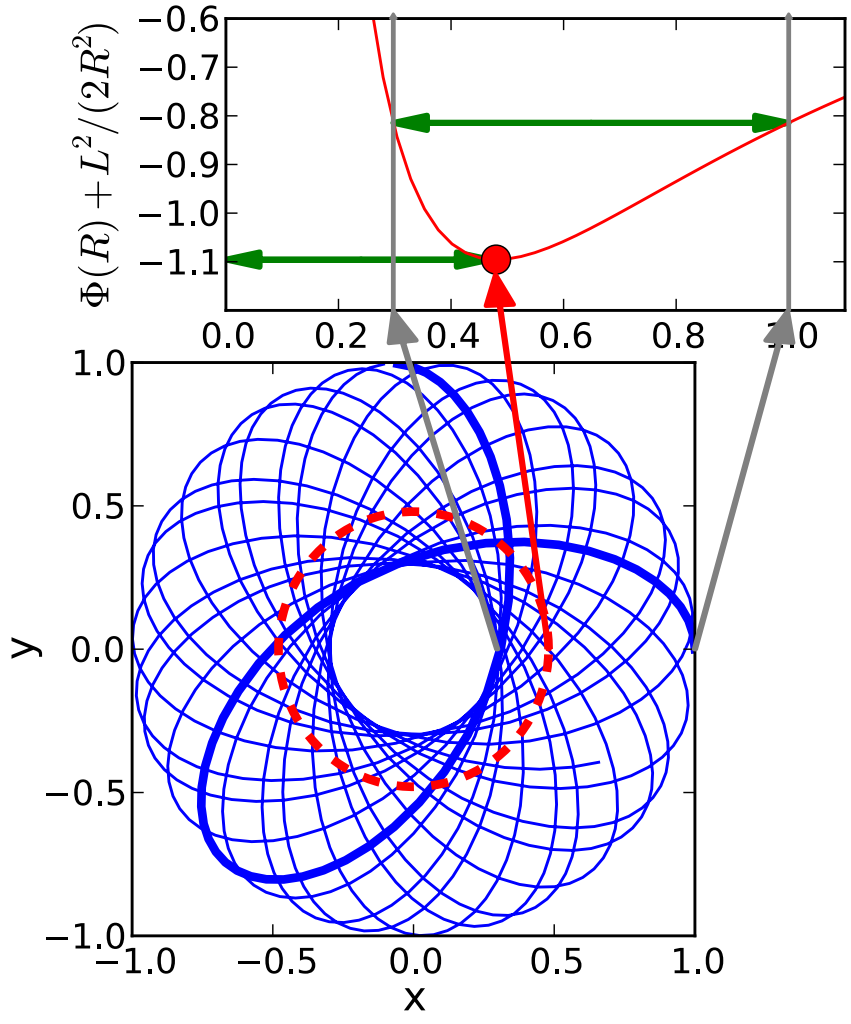
  $\frac{d\theta_i}{dt} = \frac{\partial H}{\partial J_i} \equiv \Omega_i$ (angles increase linearly with time);

  here $H(\mathbf{J})$ is the Hamilonian and $\boldsymbol{\Omega}(\mathbf{J})$ are the frequencies.

# Examples of action/angle variables

The meaning of the action/angle variables may vary for different classes of orbits, but generally describes the extent of oscillation in a particular direction.



# Pros and cons of action/angle variables

+ Most natural description of motion (angles change linearly with time); once $\mathbf{J}$ and $\boldsymbol{\Omega}$ have been found, orbit computation is trivial.
+ Possible range for each action variable is $[0..\infty)$ or $(-\infty..\infty)$, independently of the other ones (unlike $E$ and $L$, say).
+ Canonical coordinates: the volume of phase space $d^3x\, d^3v = d^3J\, d^3\theta$.
+ Actions are adiabatic invariants (are conserved under slow variation of potential).
+ Serve as a good starting point in perturbation theory.
— No general way of expressing the Hamiltonian $H \equiv \Phi(\mathbf{x}) + \frac{1}{2}\mathbf{v}^2$ in terms of actions (i.e., solving the Hamilton–Jacobi equation).
— Not easy to compute them in a general case.

+ Efficient methods for conversion between $\{\mathbf{x}, \mathbf{v}\}$ and $\{\mathbf{J}, \boldsymbol{\theta}\}$ have been developed in the last few years.

# "Classical" methods

- Spherical systems:
  two of the actions can be taken to be the *azimuthal action* $J_\phi \equiv L_z$ and the *latitudinal action* $J_\vartheta \equiv L - |L_z|$;
  the third one (the *radial action*) is given by a 1d quadrature:

$$J_r = \frac{1}{\pi} \int_{r_{\min}}^{r_{\max}} dr \, \sqrt{2[E - \Phi(r)] - L^2/r^2},$$

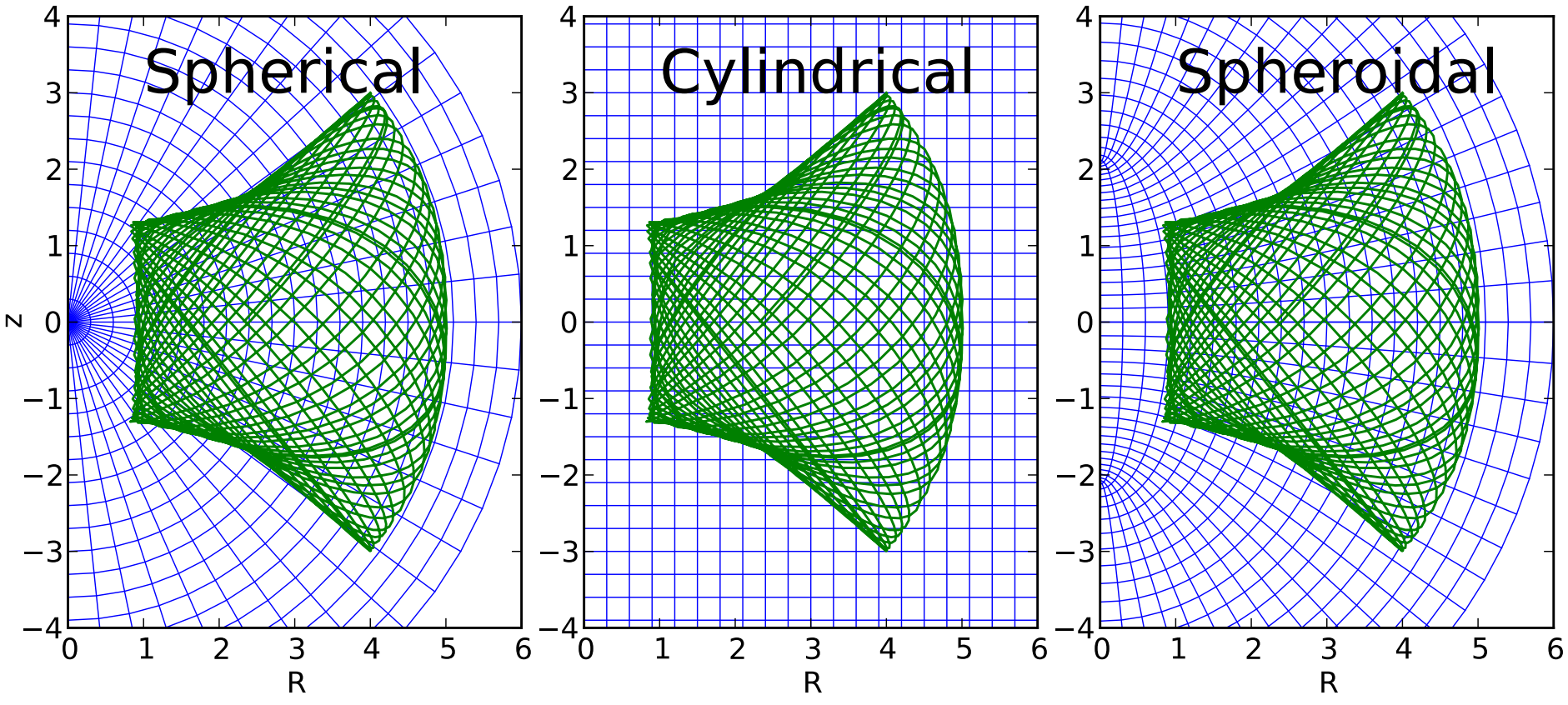
  where $r_{\min}, r_{\max}$ are the peri- and apocentre radii.
  Angles are given by 1d quadratures. For special cases (the isochrone potential, and its limiting cases – Kepler and harmonic potentials), these integrals are computed analytically.

  Note: a related concept in celestial mechanics are the Delaunay variables.

- Flattened axisymmetric systems – the **epicyclic approximation**: motion close to the disc plane is nearly separable into the in-plane motion ($J_\phi$ and $J_r$ as in the spherical case) and the vertical oscillation with a fixed energy $E_z$ in a nearly harmonic potential ($J_z$).

# State of the art: Stäckel fudge

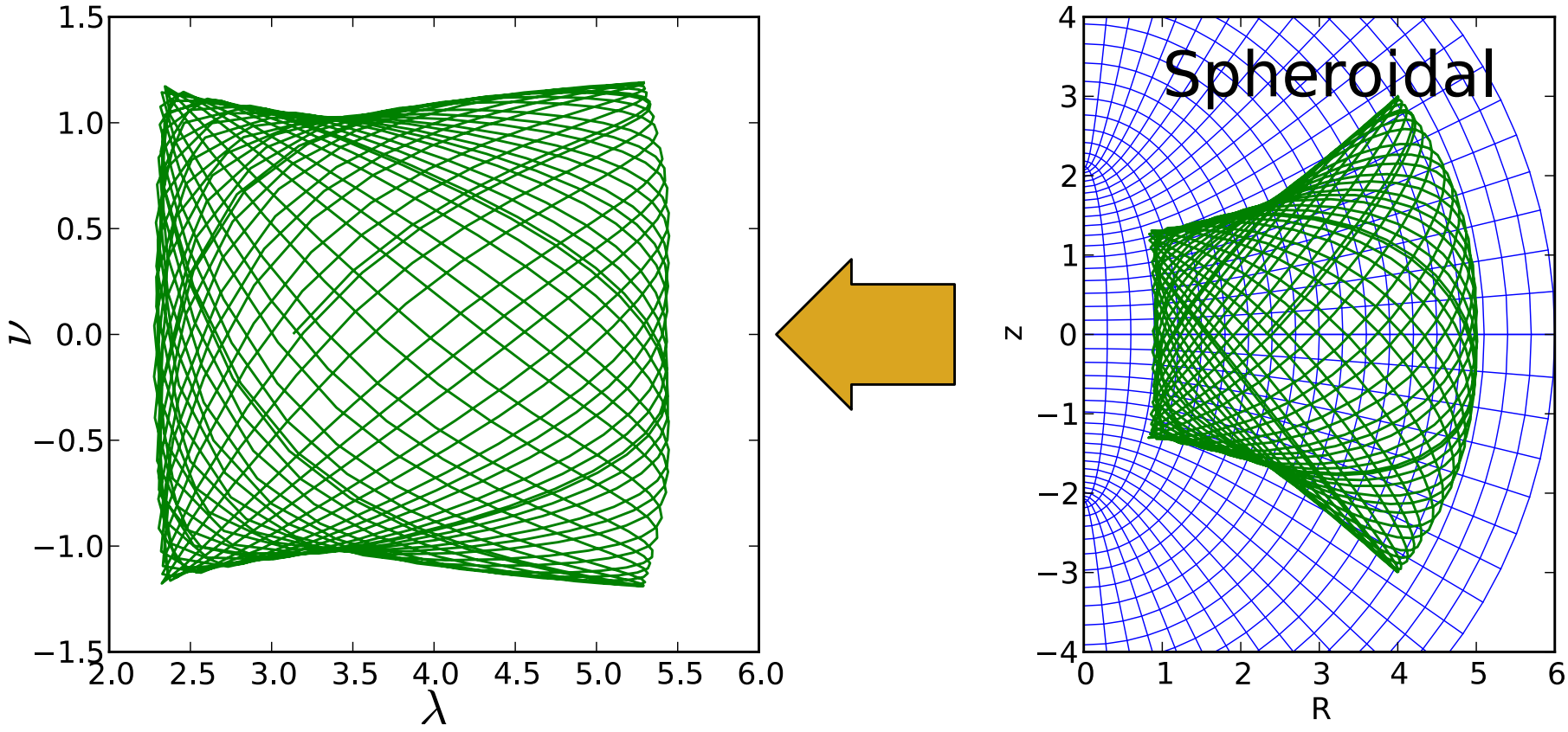
Fact: orbits in realistic axisymmetric galactic potentials are much better aligned with prolate spheroidal coordinates.



# State of the art: Stäckel fudge

Fact: orbits in realistic axisymmetric galactic potentials are much better aligned with prolate spheroidal coordinates.

One may explore the assumption that the motion is separable in these coordinates $(\lambda, \nu)$.



# Stäckel fudge (Binney 2012)

The most general form of potential that satisfies the separability condition is the Stäckel potential[1]: $\Phi(\lambda,\nu) = -\dfrac{f_1(\lambda) - f_2(\nu)}{\lambda - \nu}$.

The motion in $\lambda$ and $\nu$ directions, with canonical momenta $p_\lambda, p_\nu$, is governed by two separate equations:

$$2(\lambda - \Delta^2)\,\lambda\,p_\lambda^2 = \left[E - \frac{L_z^2}{2(\lambda - \Delta^2)}\right]\lambda - [I_3 + (\lambda - \nu)\Phi(\lambda,\nu)],$$

$$2(\nu - \Delta^2)\,\nu\,p_\nu^2 = \left[E - \frac{L_z^2}{2(\nu - \Delta^2)}\right]\nu - [I_3 + (\nu - \lambda)\Phi(\lambda,\nu)].$$

Under the approximation that the separation constant $I_3$ is indeed conserved along the orbit, this allows to compute the actions:

$$J_\lambda = \frac{1}{\pi}\int_{\lambda_{\rm min}}^{\lambda_{\rm max}} p_\lambda\,d\lambda, \quad J_\nu = \frac{1}{\pi}\int_{\nu_{\rm min}}^{\nu_{\rm max}} p_\nu\,d\nu.$$

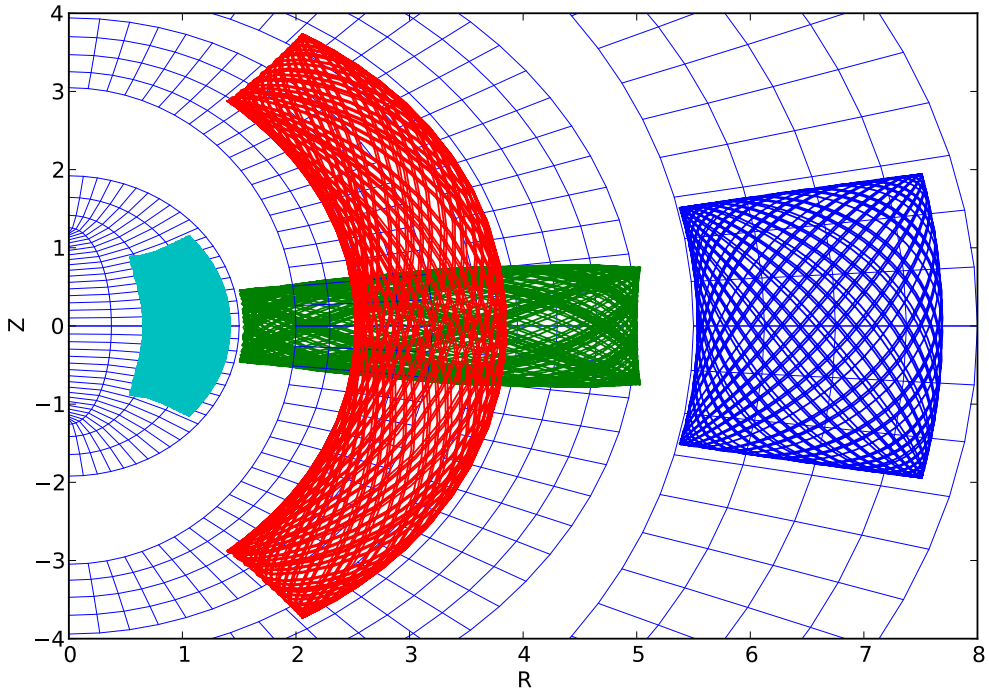
[1] Note that the potential of the Perfect Ellipsoid (de Zeeuw 1985) is of the Stäckel form, but it is only one example of a much wider class of potentials.

# Stäckel fudge in practice

A rather flexible approximation: for each orbit, we have the freedom of using two functions $f_1(\lambda)$, $f_2(\nu)$ (directly evaluated from the actual potential $\Phi(R, z)$) to describe the motion in two independent directions.

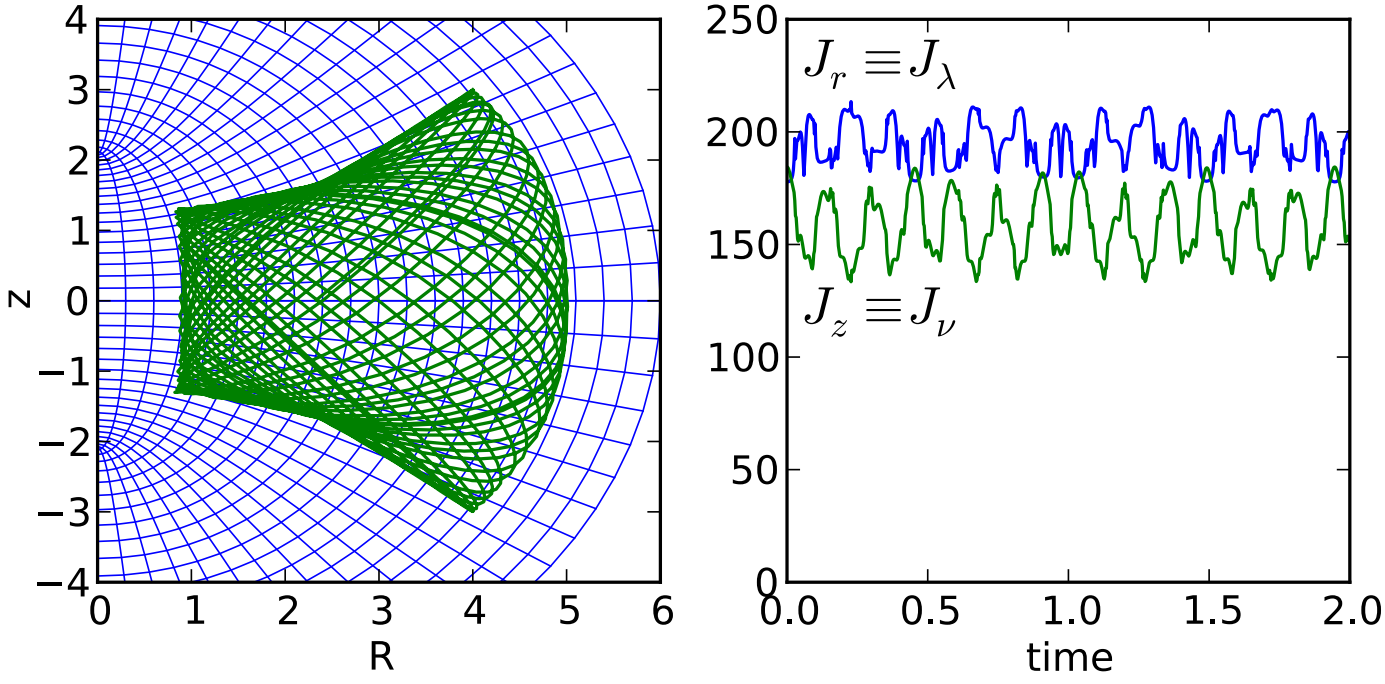
These functions are different for each orbit (implicitly depend on $E, L_z, I_3$).

Moreover, we may choose the interfocal distance $\Delta$ of the auxiliary prolate spheroidal coordinate system for each orbit independently.



# Accuracy of Stäckel fudge

Accuracy of action conservation using the Stäckel fudge:
$\lesssim 1\%$ for most disc orbits, $\lesssim 10\%$ even for high-eccentricity orbits.




But it is "uncontrollable",
accuracy cannot be systematically improved.

Can we do better?

# Other methods for action computation

**Yes**, actions offer the only **systematic** method for computing the integrals of motion in a **non-perturbative** way.

Canonical transformation between true $\{\mathbf{J}, \boldsymbol{\theta}\}$ and "toy" $\{\mathbf{J}^T, \boldsymbol{\theta}^T\}$ in some simple potential (e.g., isochrone), for which the mapping between position/velocity and action/angle coordinates is known (Torus construction – McGill&Binney 1990, McMillan&Binney 2008).

This transformation is described by a generating function $S(\mathbf{J}, \boldsymbol{\theta}^T)$, which can be expanded into Fourier series in $\boldsymbol{\theta}^T$; the accuracy of this approximation depends on the number of terms in the expansion.

A **global** map between toy and true action/angles is obtained by interpolating the coefficients of the Fourier expansion as functions of actions – once this map is constructed, the transformations $\{\mathbf{J}, \boldsymbol{\theta}\} \iff \{\mathbf{x}, \mathbf{v}\}$ are fast and accurate (except near resonances).

[work in progress...]

# Models with a prescribed distribution function

**Goal:** $f(\mathbf{J}) \implies \rho(\mathbf{x})$

given a particular expression for the distribution function $f(\mathbf{J})$, construct the corresponding self-consistent potential/density pair.

- Assume an initial guess for $\Phi(\mathbf{x})$;
- Construct the mapping $\{\mathbf{x}, \mathbf{v}\} \implies \{\mathbf{J}, \boldsymbol{\theta}\}$ in this potential;
- Compute $\rho(\mathbf{x}) = \iiint d^3v\, f\big(\mathbf{J}(\mathbf{x}, \mathbf{v})\big)$;
- Solve the Poisson equation to find new $\Phi(\mathbf{x})$;
- Repeat until convergence.

# Advantages of using actions

1. Action/angle variables are canonical $\Longrightarrow$

   the total mass of the model is computed trivially

   $$M = \int f(\mathbf{x}, \mathbf{v})\, d^3x\, d^3v = \int f(\mathbf{J})\, d^3J\, (2\pi)^3,$$

   does not depend on $\Phi$, does not change between iterations.

2. Multicomponent models:

   trivial superposition of separate $f_k(\mathbf{J})$ without changing the functional form of each component;

   addition of a new component $\Longrightarrow$
   adiabatic modification of existing density profiles
   (e.g., dark matter halo response to the formation of a baryonic disc).

3. Faster and more robust convergence ($\sim 5 - 10$ iterations).

# How to compute the potential in a general case

1. Direct integration:

$$\Phi(\mathbf{x}) = -\iiint d^3x'\,\rho(\mathbf{x}') \times \frac{G}{|\mathbf{x}-\mathbf{x}'|}.$$

3. Spherical-harmonic expansion:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \Phi_{lm}(r)\,Y_l^m(\theta,\phi),$$

$$\Phi_{lm}(r) = -\frac{4\pi G}{2l+1} \times$$
$$\times \left[ r^{-1-l}\int_0^r dr'\,\rho_{lm}(r')\,r'^{\,l+2} + r^l \int_r^{\infty} dr'\,\rho_{lm}(r')\,r'^{\,1-l}\right],$$

$$\rho_{lm}(r) = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\,\rho(r,\theta,\phi)\,Y_l^{m*}(\theta,\phi).$$

# How to compute the potential in a general case

**2.** Azimuthal-harmonic (Fourier) expansion:

$$\Phi(R,z,\phi) = \sum_{m=-\infty}^{\infty} \Phi_m(R,z)\, \mathrm{e}^{im\phi},$$

$$\rho_m(R,z) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\; \rho(R,z,\phi)\mathrm{e}^{-im\phi},$$

$$\Phi_m(R,z) = -\iint dR'\, dz'\, \rho_m(R',z') \times \Xi_m(R,z,R',z'),$$

analytic expr. for Green's function:

$$\Xi_m(R,z,R',z') \equiv \int_0^{\infty} dk\; 2\pi G\; J_m(kR)\, J_m(kR')\, \exp(-k|z-z'|) =$$

$$= \frac{2\sqrt{\pi}\,\Gamma\left(m+\frac{1}{2}\right)\; {}_2F_1\left(\frac{3}{4}+\frac{m}{2},\, \frac{1}{4}+\frac{m}{2};\, m+1;\, \xi^{-2}\right)}{\sqrt{RR'}\,(2\xi)^{m+1/2}\,\Gamma(m+1)}$$

$$\text{where } \xi \equiv \frac{R^2+R'^2+(z-z')^2}{2RR'}.$$

# How to compute the potential in a general case

1. Direct integration:

$$\Phi(\mathbf{x}) = -\iiint d^3x'\,\rho(\mathbf{x}')\times\frac{G}{|\mathbf{x}-\mathbf{x}'|}.$$

2. Azimuthal harmonic expansion:

$$\Phi(R,z,\phi) = \sum_{m=-\infty}^{\infty}\Phi_m(R,z)\,\mathrm{e}^{im\phi}.$$

3. Spherical harmonic expansion:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\Phi_{lm}(r)\,Y_l^m(\theta,\phi).$$

interpolated functions

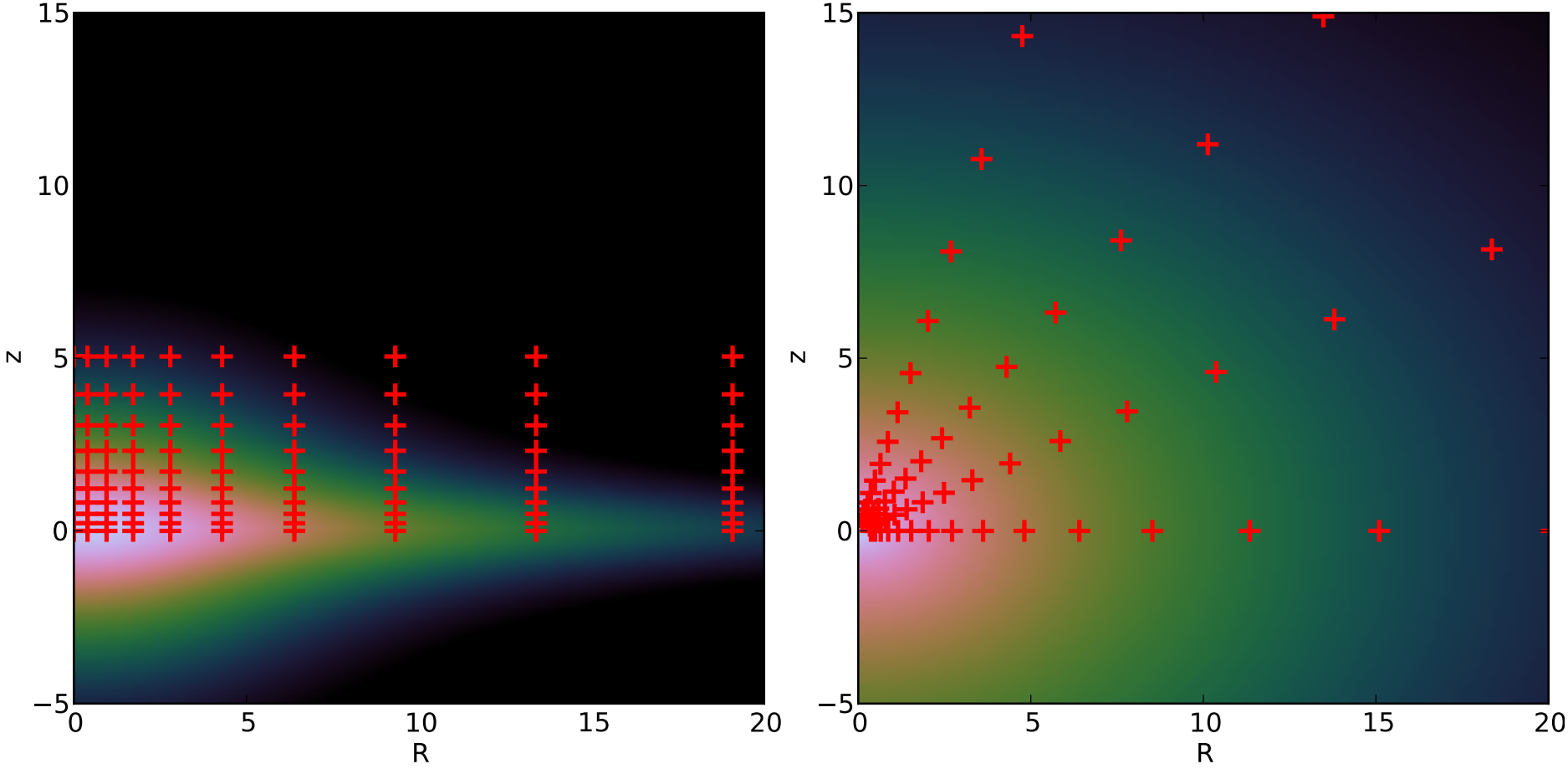
4. Basis-set expansion:

$$\Phi(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\Phi_{nlm}\,A_{nl}(r)\,Y_l^m(\theta,\phi).$$

(example: self-consistent field method of Hernquist&Ostriker 1992)

# Two types of potential approximations used in models

- for disc-like components – azimuthal-harmonic expansion;
- for spheroidal components – spherical-harmonic expansion.



# Gravitational potential extracted from N-body models

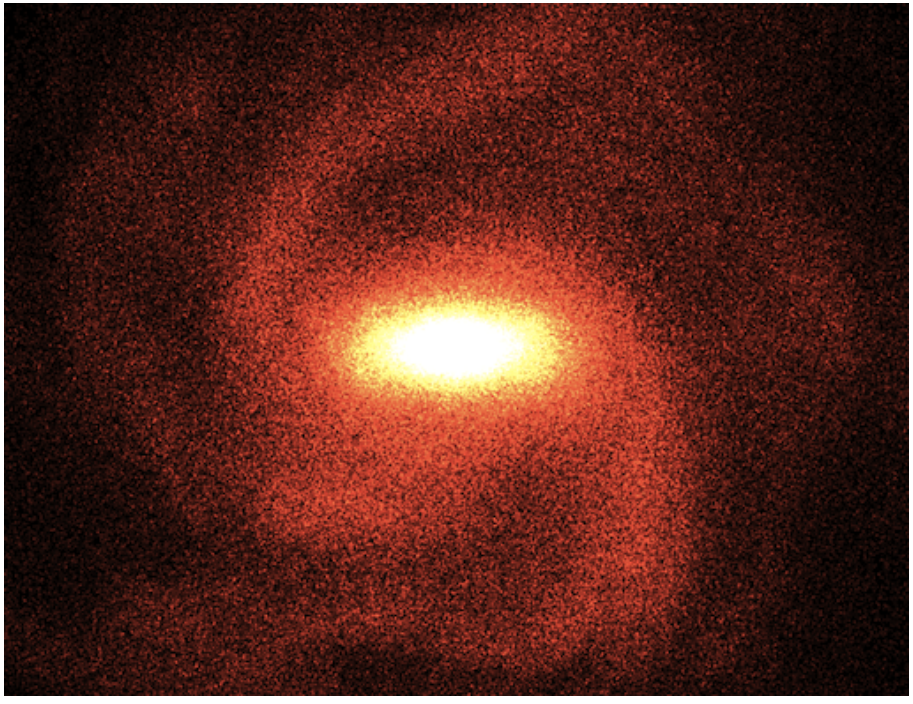
The spherical-harmonic and azimuthal-harmonic potential approximations can also be constructed from *N*-body models.

Advantages:

fast evaluation, smooth forces, suitable for orbit analysis.

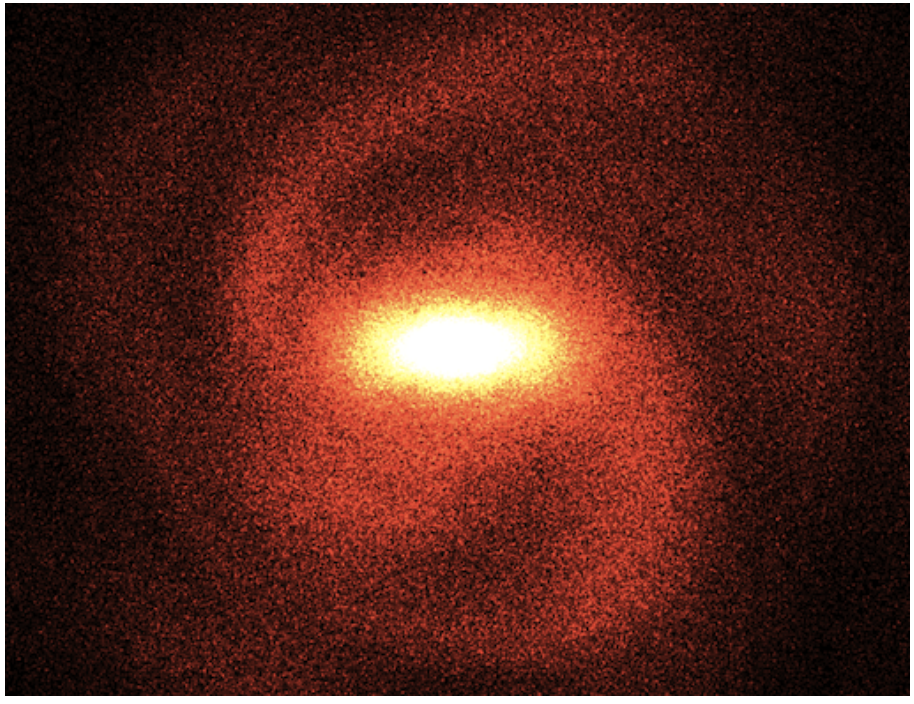
Real *N*-body model

(from Roca-Fabrega et al. 2013, 2014)



Potential approximation

(suitable for test-particle integrations, e.g. Romero-Gomez et al. 2011)



# Models with prescribed density profile

**Goal:** $\rho(\mathbf{x}) \implies f(\mathcal{I})$

construct a self-consistent model for the given density profile (possibly with additional kinematic constraints).

Need to somehow invert the integral equation

$$\rho(\mathbf{x}) = \iiint d^3v \; f\big(\mathcal{I}(\mathbf{x},\mathbf{v})\big)$$

Variants of methods:

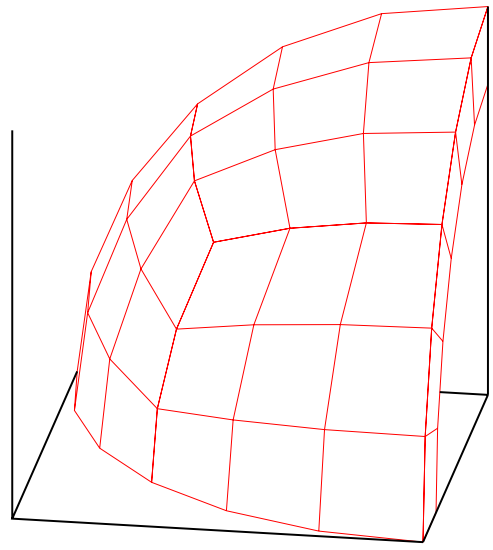
- Based on $N$-body models: made-to-measure and alike...
- Based on orbits: Schwarzschild's orbit-superposition method.
- Based on distribution-function "building blocks": this work.

# Schwarzschild's orbit-superposition method

Discretize both the density profile and the distribution function:

$\rho(\mathbf{x}) \implies$ cells of a spatial grid;
mass of each cell is

$$m_c = \iiint_{\mathbf{x}\in V_c} \rho(\mathbf{x})\, d^3x$$



$f(\mathcal{I}) \implies$ collection of orbits:

$$f(\mathcal{I}) = \sum_{k=1}^{N_{\rm orb}} w_k\ \delta(\mathcal{I} - \mathcal{I}_k)$$

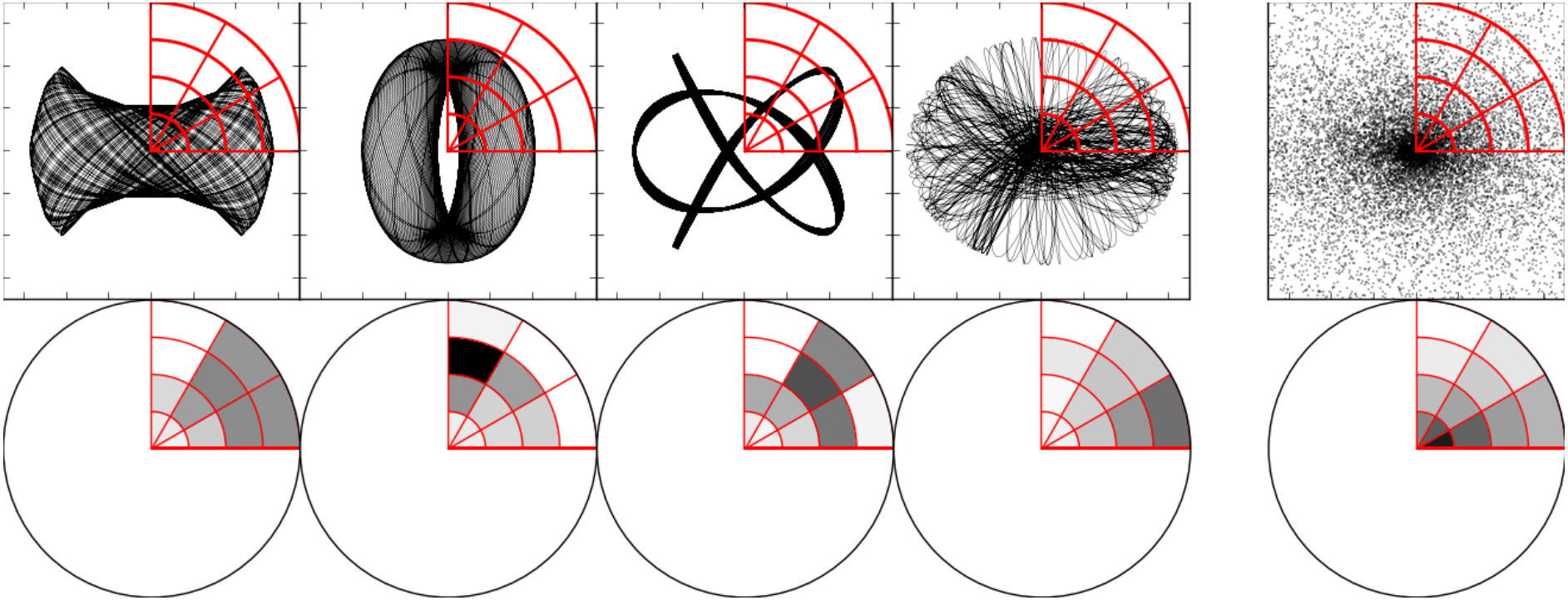
each orbit is a delta-function in the space of integrals

adjustable weight of each orbit

# Schwarzschild's orbit-superposition method

orbits in the model

target density



discretized orbit density

(fraction of time $t_{kc}$ that $k$-th orbit spends in $c$-th cell)

discretized density

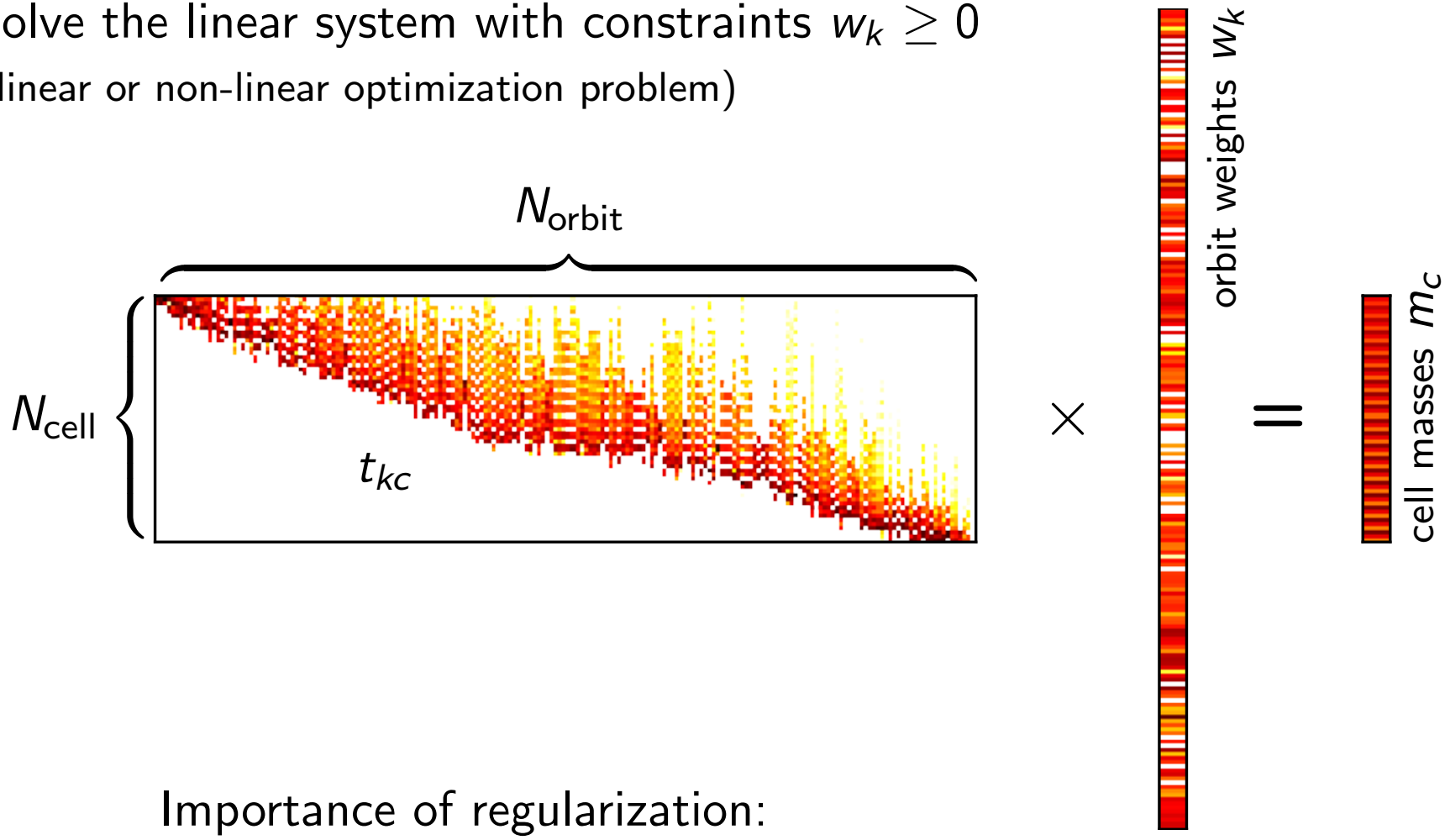
(mass $m_c$ in grid cells)

For each $c$-th cell we require $\sum_k w_k\, t_{kc} = m_c$, where $w_k \geq 0$ is orbit weight

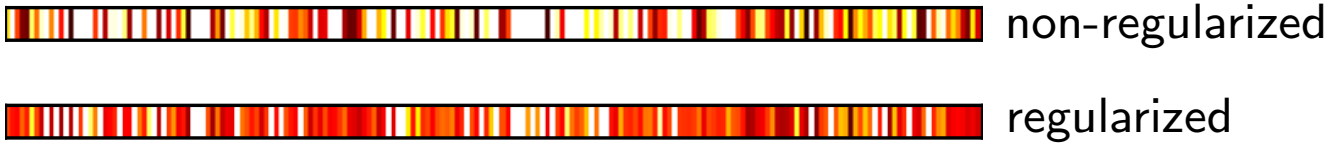
# Schwarzschild's orbit-superposition method

Solve the linear system with constraints $w_k \geq 0$

(linear or non-linear optimization problem)




Importance of regularization:



# Building blocks for distribution function

Same idea: invert the integral equation

$$\rho(\mathbf{x}) = \iiint d^3v \; f\big(\mathcal{I}(\mathbf{x},\mathbf{v})\big)$$

by decomposing $f$ into a sum of building blocks (basis functions):

$$f(\mathcal{I}) = \sum_{k=1}^{N_{\text{basis}}} w_k \, f_k(\mathcal{I}),$$

computing the projections of all basis functions at a grid of points $\mathbf{x}_c$

$$\rho_{kc} \equiv \iiint d^3v \; f_k\big(\mathcal{I}(\mathbf{x}_c,\mathbf{v})\big),$$

and solving the optimization problem

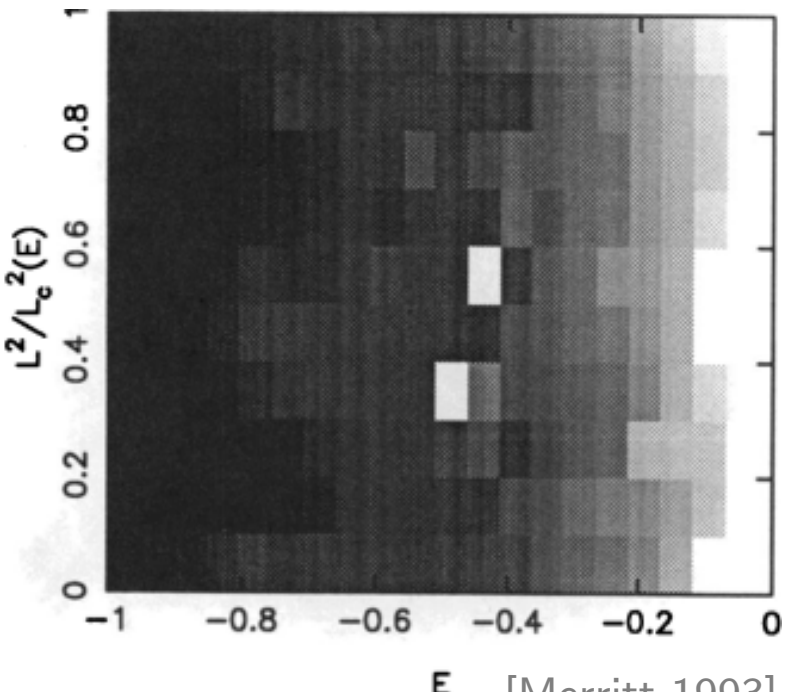
$$\rho_c \equiv \rho(\mathbf{x}_c) = \sum_{k=1}^{N_{\text{basis}}} w_k \, \rho_{kc} \,, \quad w_k \geq 0 \,, \quad c = 1..N_{\text{constraint}},$$

to find the weights of basis functions $w_k$.

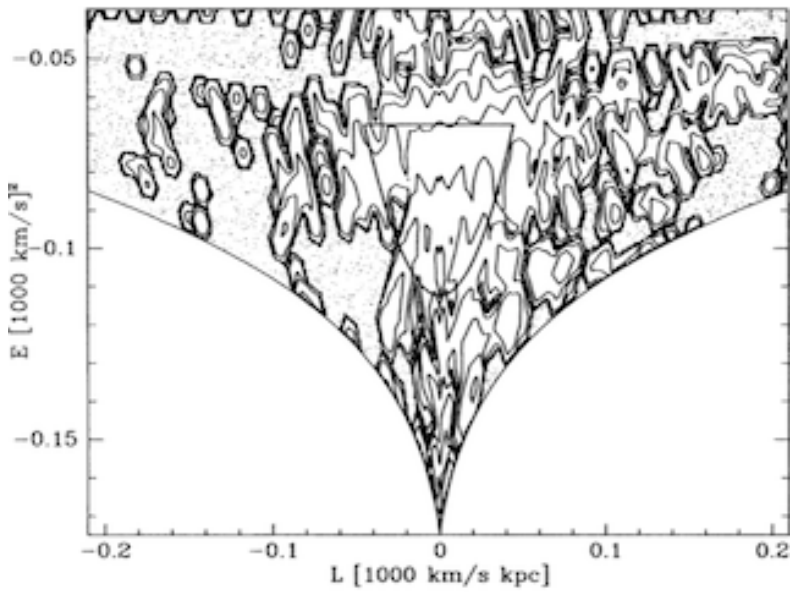
# Building blocks for distribution function

Similar approaches suggested previously:

- Dejonghe(1989), Merritt&Saha(1993): $f_k(E, L)$ as Fricke components $|E|^{\alpha}\, L^{-2\beta}$;
- Merritt(1993,1996): histograms ($\Pi$-shaped blocks) for $f(E, L)$ or $f(E, L_z)$;
- Kuijken(1994), Pichon&Thiébaut(1998): bilinear interpolation for $f(E, L_z)$;
- Dehnen&Gerhard(1994): Chebyshev polynomial basis for $f(E, L_z)$;
- Magorrian(2014): superposition of multivariate Gaussian 'blobs' for $f(E, L)$.

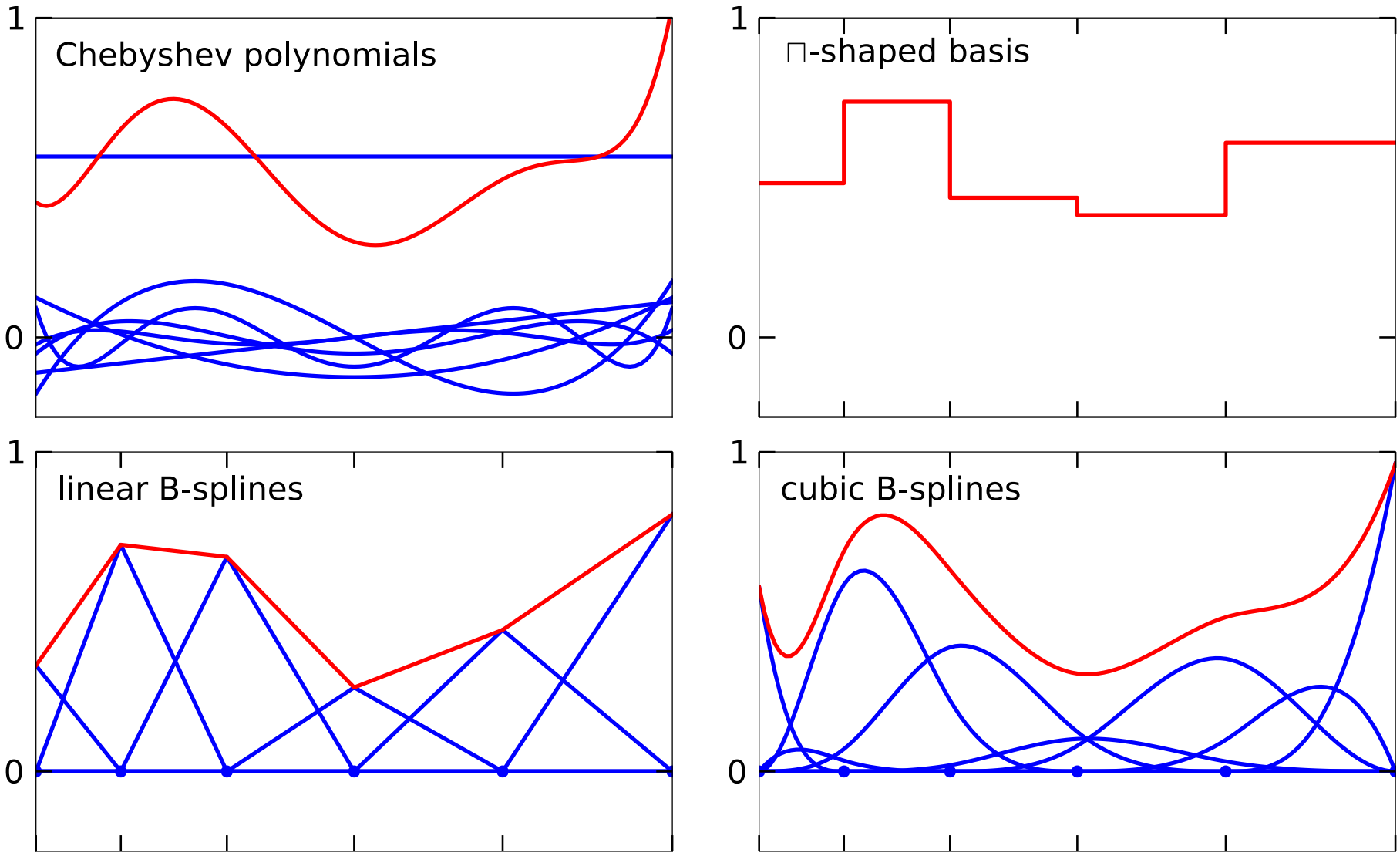



[Merritt 1993]




[Kuijken 1994]

# Building blocks for interpolation




B-splines of degree $N$:

flexible choice of grid points, locality, smoothness (increases with $N$), nonnegativity.

# Models with non-parametric distribution function

- $f(\mathbf{J})$ represented as an interpolated 3d function in action space (tensor product of 1d B-splines);
- weights of basis functions found by solving a linear or quadratic optimization problem (constraints: values of density at a 3d grid in space);
- smoothness: choice of degree of B-splines;
- regularization: minimum-curvature condition for the 3d interpolant (roughness penalty);
- possibility of determining $f(\mathbf{J})$ from an $N$-body model or from discrete observational points, using maximum penalized likelihood method (work in progress..).

# Advantages of models based on distribution function

- Clear physical meaning
  (localized structures in the space of integrals of motion);
- Easy to compare different models
  (how to compare two $N$-body or $N$-orbit models?);
- Easy to compare models to discrete observational data;
- Easy to sample particles from the distribution function
  (convert to an $N$-body model);
- Stability analysis
  (perturbation theory most naturally formulated in terms of actions);

## Caveats:

- Implicitly rely on the integrability of the potential,
  ignore the presence of resonant orbit families;
- So far implemented only for axisymmetric models
  (not a fundamental limitation).

# AGAMA library – Action-based galaxy modeling architecture

- Extensive collection of gravitational potential models
  (analytic profiles, azimuthal- and spherical-harmonic expansions);
- Conversion to/from action/angle variables
  (fast and accurate method for spherical potentials, Stäckel fudge for axisymmetric potentials, torus mapping);
- Action-based distribution functions; generation of $N$-body models and determination of best-fit parameters of DF and potential;
- Self-consistent multicomponent models with action-based DFs:
  (iterative method for $f(\mathbf{J}) \Rightarrow \rho(\mathbf{x})$, non-parametric DF recovery $\rho(\mathbf{x}) \Rightarrow f(\mathbf{J})$);
- Efficient and carefully designed C++ implementation, examples, Python interface, compatibility with other software such as galpy;

`https://github.com/GalacticDynamics-Oxford/Agama`

# Conclusions

- Advantage of models based on distribution functions;
- Advantage of actions as arguments of distribution functions;
- Two approaches for construction of self-consistent models;
- Software available for the community.

THANK YOU!