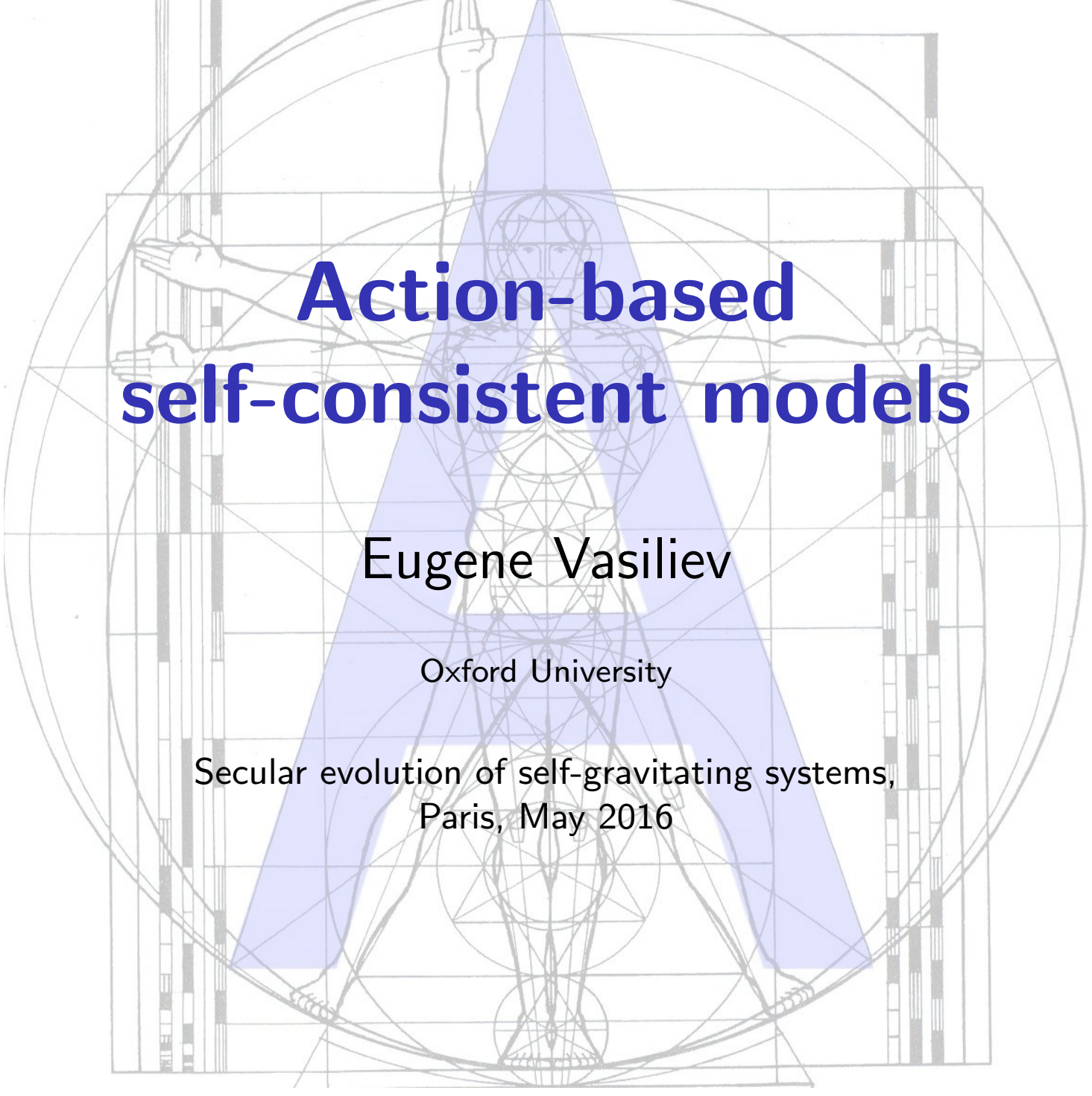


Action-based self-consistent models

Eugene Vasiliev

Oxford University

Secular evolution of self-gravitating systems,
Paris, May 2016

Fundamental equations

gravitational potential

distribution function

1. Collisionless Boltzmann equation:

$$\cancel{\frac{\partial f}{\partial t}} + \mathbf{v}\,\frac{\partial f}{\partial \mathbf{x}} - \frac{\partial \Phi}{\partial \mathbf{x}}\,\frac{\partial f}{\partial \mathbf{v}} = 0.$$

(Assumption: a galaxy is a collisionless system in a steady state)

2. Poisson equation:

density

$$\nabla^2 \Phi(\mathbf{x}) = 4\pi\, G\, \rho(\mathbf{x}).$$

(Assumption: Newtonian gravity)

3. The link:

$$\rho(\mathbf{x}) = \iiint d^3v\, f(\mathbf{x}, \mathbf{v}).$$

(Assumption: self-consistency)

Ways of solving the equations

1. Collisionless Boltzmann equation:

   Jeans theorem states that the DF may depend only on the integrals of motion

$$f = f\big(\mathcal{I}(\mathbf{x},\mathbf{v})\big)\ , \quad \mathcal{I} = \{E, L, \dots\}.$$

   depend on the potential Φ

2. Poisson equation:

$$\Phi(\mathbf{x}) = -\iiint d^3x'\, G\, \rho(\mathbf{x}') \times \frac{1}{|\mathbf{x}-\mathbf{x}'|}.$$

3. The link:

$$\rho(\mathbf{x}) = \iiint d^3v\, f\big(\mathcal{I}(\mathbf{x},\mathbf{v})\big).$$

Self-consistent models of spherical systems

- Integrals of motion:

  energy $E = \Phi(r) + v^2/2$, angular momentum $L = |\mathbf{x} \times \mathbf{v}|$.

- Distribution function:

  $f(E)$ (isotropic) or $f(E, L)$ (general anisotropic).

Two possible approaches $\quad f \implies \rho \quad$ or $\quad \rho \implies f$:

1. From f to ρ:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi(r)}{dr}\right) = 4\pi G \iiint d^3v\, f\big(\Phi(r) + v^2/2,\ |\mathbf{x} \times \mathbf{v}|\big)$$

(second-order integro-differential equation for $\Phi(r)$).

Examples: Plummer, King, lowered isothermal models, etc.

Self-consistent models of spherical systems

2. From ρ to f:

- Compute $\Phi(r)$ and its inverse $r(\Phi)$:

$$\Phi(\mathbf{x}) \equiv \Phi(r) = -4\pi G \left[r^{-1} \int_0^r dr'\, \rho(r')\, r'^{\,2} + \int_r^\infty dr'\, \rho(r')\, r'^{\,-1} \right].$$

- Invert the integral equation

$$\rho(r) = 4\pi G \iiint d^3v\, f\big(\Phi(r) + v^2/2,\ |\mathbf{x}\times\mathbf{v}|\big) \quad \Longrightarrow$$

$$f(E) = \frac{d}{dE} \int_E^0 d\Phi\, \frac{d\rho\big(r(\Phi)\big)}{d\Phi}\, \frac{1}{4\pi\sqrt{2[E-\Phi(r)]}}$$

(Eddington inversion), or its generalizations for anisotropic $f\big[Q(E,L)\big]$ (Ossipkov–Merritt, Cuddeford, etc.)
Examples: isochrone, Hernquist model, etc.

Axisymmetric systems

- Integrals of motion $\mathcal{I}$:
  $E = \Phi(R,z) + v^2/2, \quad L_z = R\,v_\phi,$
  third integral $I_3 = ?$ (known analytically in special cases).
- Distribution function:
  $f(E, L_z)$ (two-integral) or $f(E, L_z, I_3)$ (general three-integral).

Two approaches – ?

$$\nabla^2\Phi(R,z) = 4\pi G\rho(R,z) = 4\pi G \iiint d^3v\; f\big(\mathcal{I}(R,z,\mathbf{v})\big)$$

No simple way to solve the integro-partial-differential equation, or to invert it to obtain $f(\mathcal{I})$ from $\rho(R,z)$

(apart from the contour integral method of Hunter&Qian 1993).

Models with a prescribed distribution function

Goal: $f(\mathbf{J}) \implies \rho(\mathbf{x})$

given a particular expression for the distribution function $f(\mathbf{J})$, construct the corresponding self-consistent potential/density pair.
Method: Iterative approach [e.g., Prendergast&Tomer 1970], using actions as the integrals of motion.

- Assume an initial guess for $\Phi(\mathbf{x})$;
- Construct the mapping $\{\mathbf{x}, \mathbf{v}\} \implies \{\mathbf{J}, \boldsymbol{\theta}\}$ in this potential;
- Compute $\rho(\mathbf{x}) = \iiint d^3v\, f\big(\mathbf{J}(\mathbf{x}, \mathbf{v})\big)$;
- Solve the Poisson equation to find new $\Phi(\mathbf{x})$;
- Repeat until convergence.

Advantages of using actions

1. Action/angle variables are canonical $\Longrightarrow$

   the total mass of the model is computed trivially

   $$M = \int f(\mathbf{x}, \mathbf{v})\, d^3x\, d^3v = \int f(\mathbf{J})\, d^3J\, (2\pi)^3,$$

   does not depend on Φ, does not change between iterations.

2. Multicomponent models:

   trivial superposition of separate $f_k(\mathbf{J})$ without changing the functional form of each component;

   addition of a new component $\Longrightarrow$
   adiabatic modification of existing density profiles
   (e.g., dark matter halo response to the formation of a baryonic disc).

3. Faster and more robust convergence ($\sim 5 - 10$ iterations).

Computing the actions: "Classical" methods

- Spherical systems:
  two of the actions can be taken to be the *azimuthal action* $J_\phi \equiv L_z$ and the *latitudinal action* $J_\vartheta \equiv L - |L_z|$;
  the third one (the *radial action*) is given by a 1d quadrature:

$$J_r = \frac{1}{\pi} \int_{r_{\min}}^{r_{\max}} dr \, \sqrt{2[E - \Phi(r)] - L^2/r^2},$$

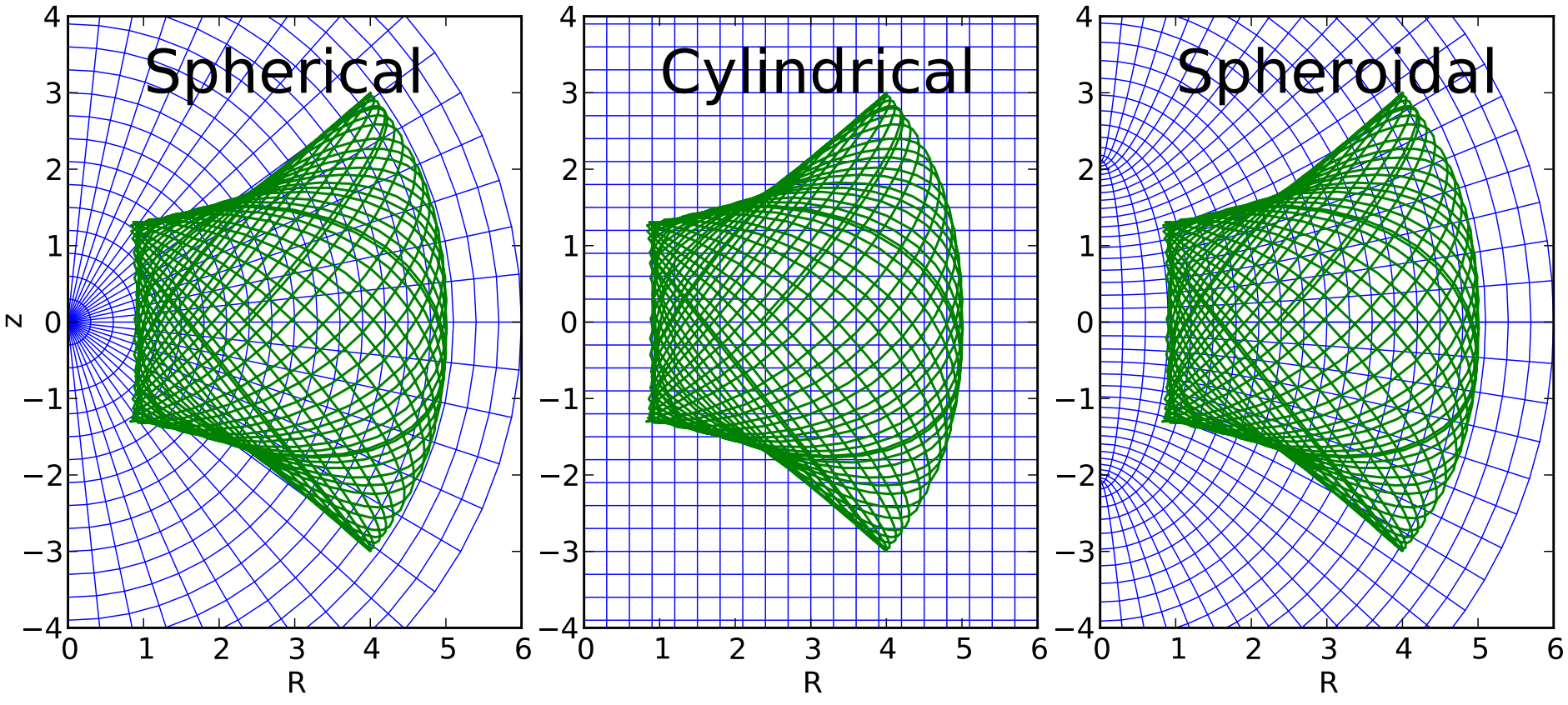
  where $r_{\min}, r_{\max}$ are the peri- and apocentre radii.
  Angles are given by 1d quadratures. For special cases (the isochrone potential, and its limiting cases – Kepler and harmonic potentials), these integrals are computed analytically.

  Note: a related concept in celestial mechanics are the Delaunay variables.

- Flattened axisymmetric systems – the **epicyclic approximation**: motion close to the disc plane is nearly separable into the in-plane motion (J_ϕ and J_r as in the spherical case) and the vertical oscillation with a fixed energy E_z in a nearly harmonic potential (J_z).

State of the art: Stäckel fudge

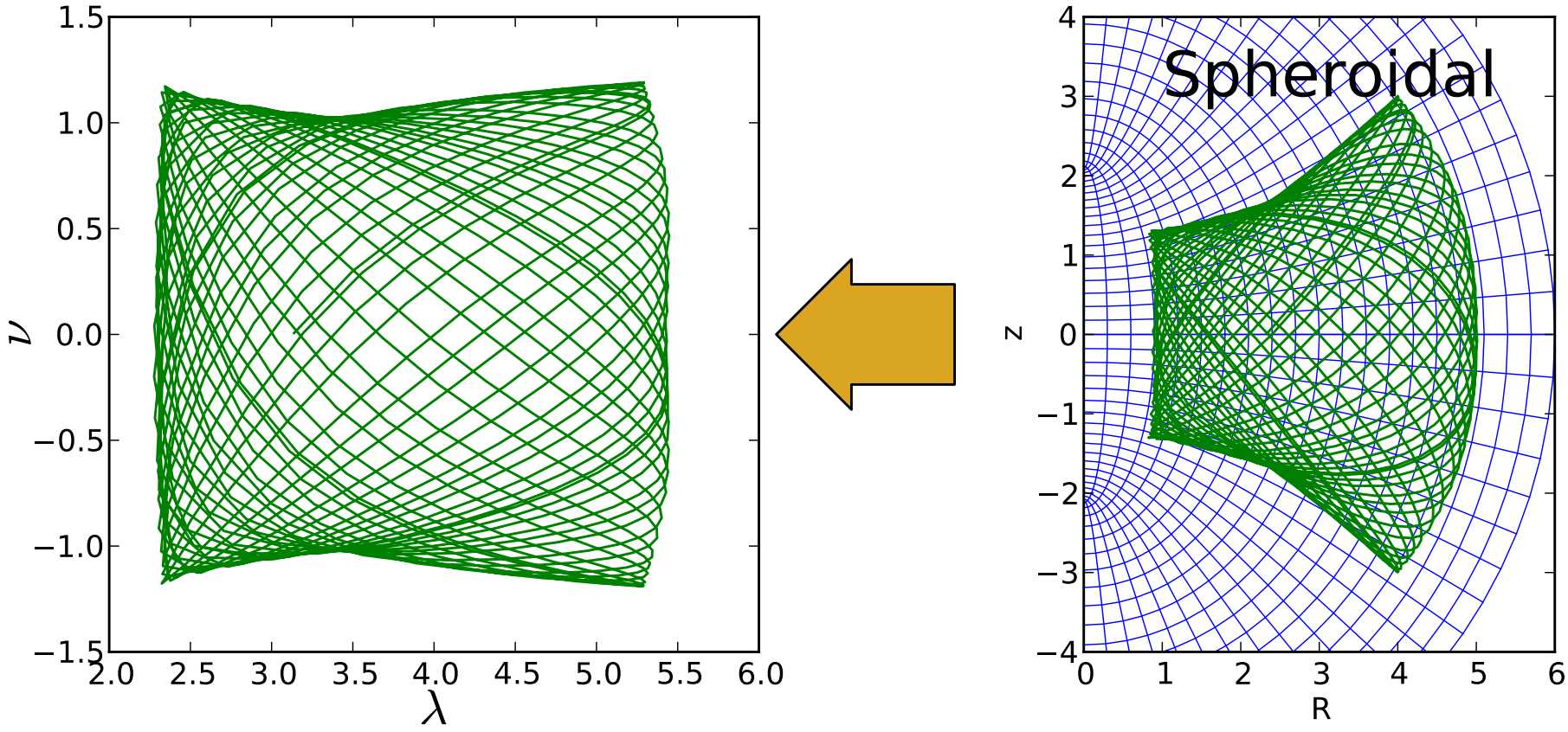
Fact: orbits in realistic axisymmetric galactic potentials are much better aligned with prolate spheroidal coordinates.



State of the art: Stäckel fudge

Fact: orbits in realistic axisymmetric galactic potentials are much better aligned with prolate spheroidal coordinates.

One may explore the assumption that the motion is separable in these coordinates (λ, ν).



Stäckel fudge (Binney 2012)

The most general form of potential that satisfies the separability condition is the Stäckel potential[1]: $\Phi(\lambda,\nu) = -\dfrac{f_1(\lambda) - f_2(\nu)}{\lambda - \nu}$.

The motion in λ and ν directions, with canonical momenta p_λ, p_ν, is governed by two separate equations:

$$2(\lambda - \Delta^2)\,\lambda\,p_\lambda^2 = \left[E - \frac{L_z^2}{2(\lambda - \Delta^2)}\right]\lambda - [I_3 + (\lambda - \nu)\Phi(\lambda,\nu)],$$

$$2(\nu - \Delta^2)\,\nu\,p_\nu^2 = \left[E - \frac{L_z^2}{2(\nu - \Delta^2)}\right]\nu - [I_3 + (\nu - \lambda)\Phi(\lambda,\nu)].$$

Under the approximation that the separation constant I_3 is indeed conserved along the orbit, this allows to compute the actions:

$$J_\lambda = \frac{1}{\pi}\int_{\lambda_{\rm min}}^{\lambda_{\rm max}} p_\lambda\,d\lambda, \quad J_\nu = \frac{1}{\pi}\int_{\nu_{\rm min}}^{\nu_{\rm max}} p_\nu\,d\nu.$$

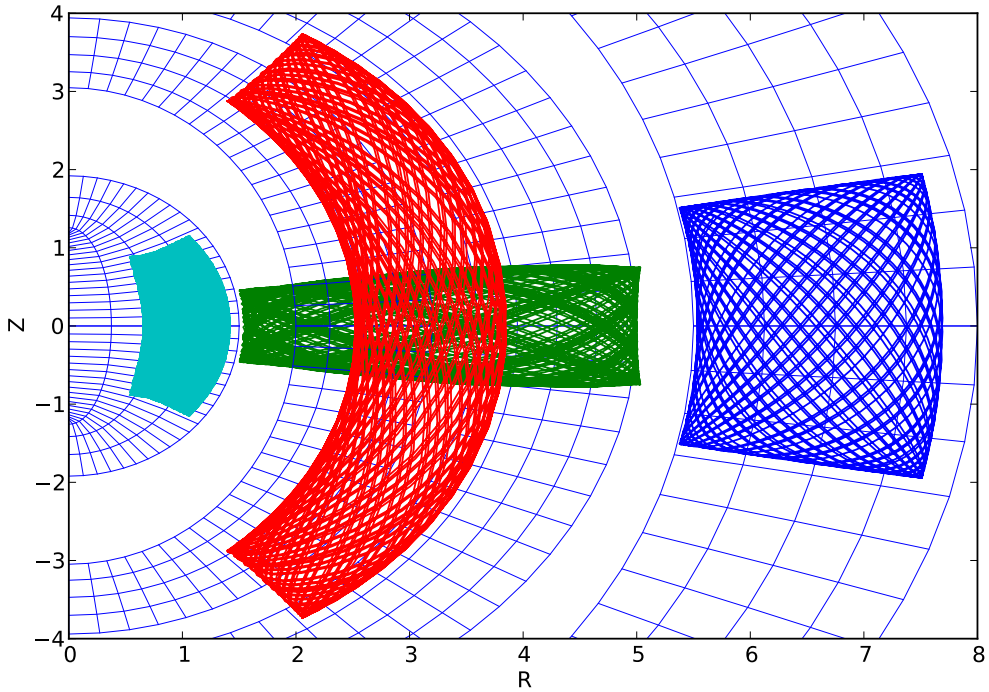
[1] Note that the potential of the Perfect Ellipsoid (de Zeeuw 1985) is of the Stäckel form, but it is only one example of a much wider class of potentials.

Stäckel fudge in practice

A rather flexible approximation: for each orbit, we have the freedom of using two functions $f_1(\lambda)$, $f_2(\nu)$ (directly evaluated from the actual potential $\Phi(R, z)$) to describe the motion in two independent directions.

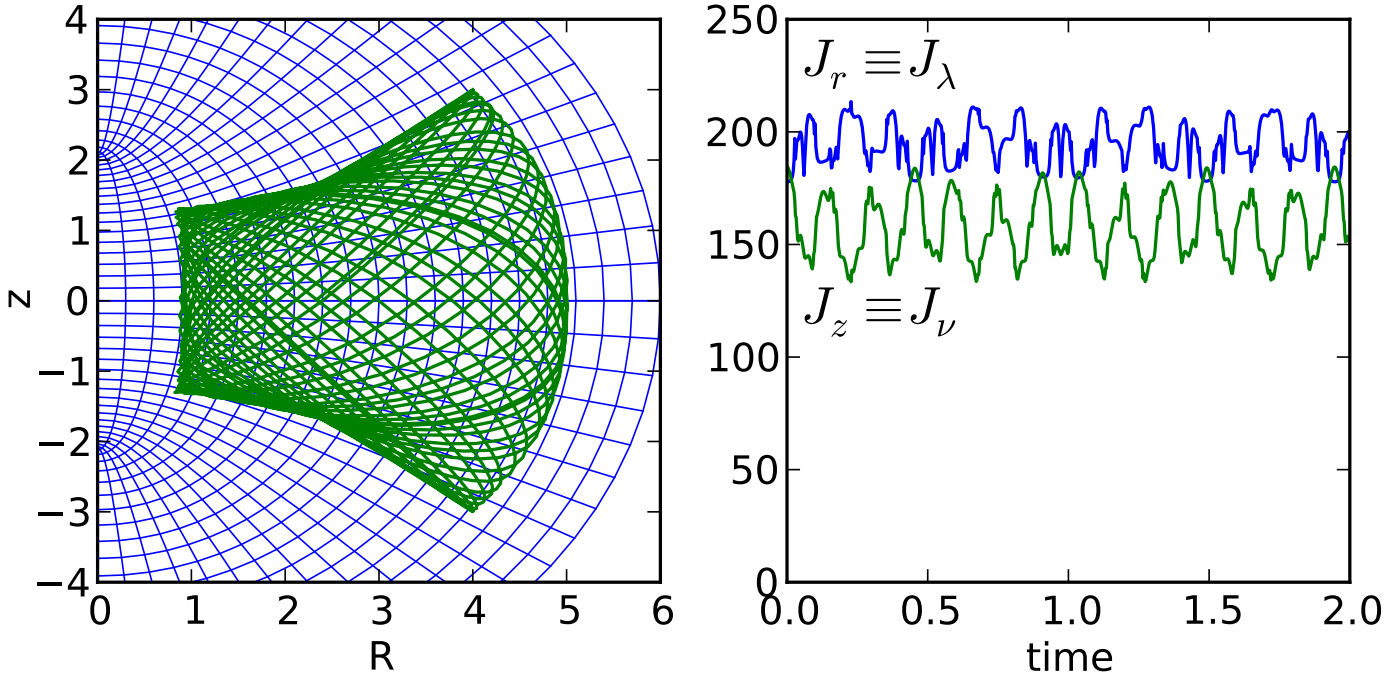
These functions are different for each orbit (implicitly depend on E, L_z, I_3).

Moreover, we may choose the interfocal distance Δ of the auxiliary prolate spheroidal coordinate system for each orbit independently.



Accuracy of Stäckel fudge

Accuracy of action conservation using the Stäckel fudge:
$\lesssim 1\%$ for most disc orbits, $\lesssim 10\%$ even for high-eccentricity orbits.



But it is "uncontrollable",
accuracy cannot be systematically improved.

Can we do better?

Other methods for action computation

Yes, actions offer the only **systematic** method for computing the integrals of motion in a **non-perturbative** way.

Canonical transformation between true $\{\mathbf{J}, \boldsymbol{\theta}\}$ and "toy" $\{\mathbf{J}^T, \boldsymbol{\theta}^T\}$ in some simple potential (e.g., isochrone), for which the mapping between position/velocity and action/angle coordinates is known (Torus construction – McGill&Binney 1990, McMillan&Binney 2008).

This transformation is described by a generating function $S(\mathbf{J}, \boldsymbol{\theta}^T)$, which can be expanded into Fourier series in $\boldsymbol{\theta}^T$; the accuracy of this approximation depends on the number of terms in the expansion.

A **global** map between toy and true action/angles is obtained by interpolating the coefficients of the Fourier expansion as functions of actions – once this map is constructed, the transformations $\{\mathbf{J}, \boldsymbol{\theta}\} \iff \{\mathbf{x}, \mathbf{v}\}$ are fast and accurate (except near resonances).

[work in progress...]

How to compute the potential in a general case

1. Direct integration:

$$\Phi(\mathbf{x}) = -\iiint d^3x'\,\rho(\mathbf{x}') \times \frac{G}{|\mathbf{x}-\mathbf{x}'|}.$$

3. Spherical-harmonic expansion:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \Phi_{lm}(r)\, Y_l^m(\theta,\phi),$$

$$\Phi_{lm}(r) = -\frac{4\pi G}{2l+1} \times$$
$$\times \left[r^{-1-l}\int_0^r dr'\,\rho_{lm}(r')\,r'^{\,l+2} + r^l \int_r^{\infty} dr'\,\rho_{lm}(r')\,r'^{\,1-l}\right],$$

$$\rho_{lm}(r) = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\,\rho(r,\theta,\phi)\,Y_l^{m*}(\theta,\phi).$$

How to compute the potential in a general case

2. Azimuthal-harmonic (Fourier) expansion:

$$\Phi(R,z,\phi) = \sum_{m=-\infty}^{\infty} \Phi_m(R,z)\,\mathrm{e}^{im\phi},$$

$$\rho_m(R,z) = \frac{1}{2\pi}\int_0^{2\pi} d\phi\;\rho(R,z,\phi)\mathrm{e}^{-im\phi},$$

$$\Phi_m(R,z) = -\iint dR'\,dz'\,\rho_m(R',z')\times \Xi_m(R,z,R',z'),$$

analytic expr. for Green's function:

$$\Xi_m(R,z,R',z') \equiv \int_0^{\infty} dk\;2\pi G\;J_m(kR)\,J_m(kR')\;\exp(-k|z-z'|) =$$

$$= \frac{2\sqrt{\pi}\,\Gamma\left(m+\frac{1}{2}\right)\;{}_2F_1\left(\frac{3}{4}+\frac{m}{2},\,\frac{1}{4}+\frac{m}{2};\,m+1;\,\xi^{-2}\right)}{\sqrt{RR'}\,(2\xi)^{m+1/2}\,\Gamma(m+1)}$$

$$\text{where } \xi \equiv \frac{R^2+R'^2+(z-z')^2}{2RR'}.$$

How to compute the potential in a general case

1. Direct integration:

$$\Phi(\mathbf{x}) = -\iiint d^3x'\,\rho(\mathbf{x}')\times\frac{G}{|\mathbf{x}-\mathbf{x}'|}.$$

2. Azimuthal harmonic expansion:

$$\Phi(R,z,\phi) = \sum_{m=-\infty}^{\infty}\Phi_m(R,z)\,\mathrm{e}^{im\phi}.$$

3. Spherical harmonic expansion:

$$\Phi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\Phi_{lm}(r)\,Y_l^m(\theta,\phi).$$

interpolated functions

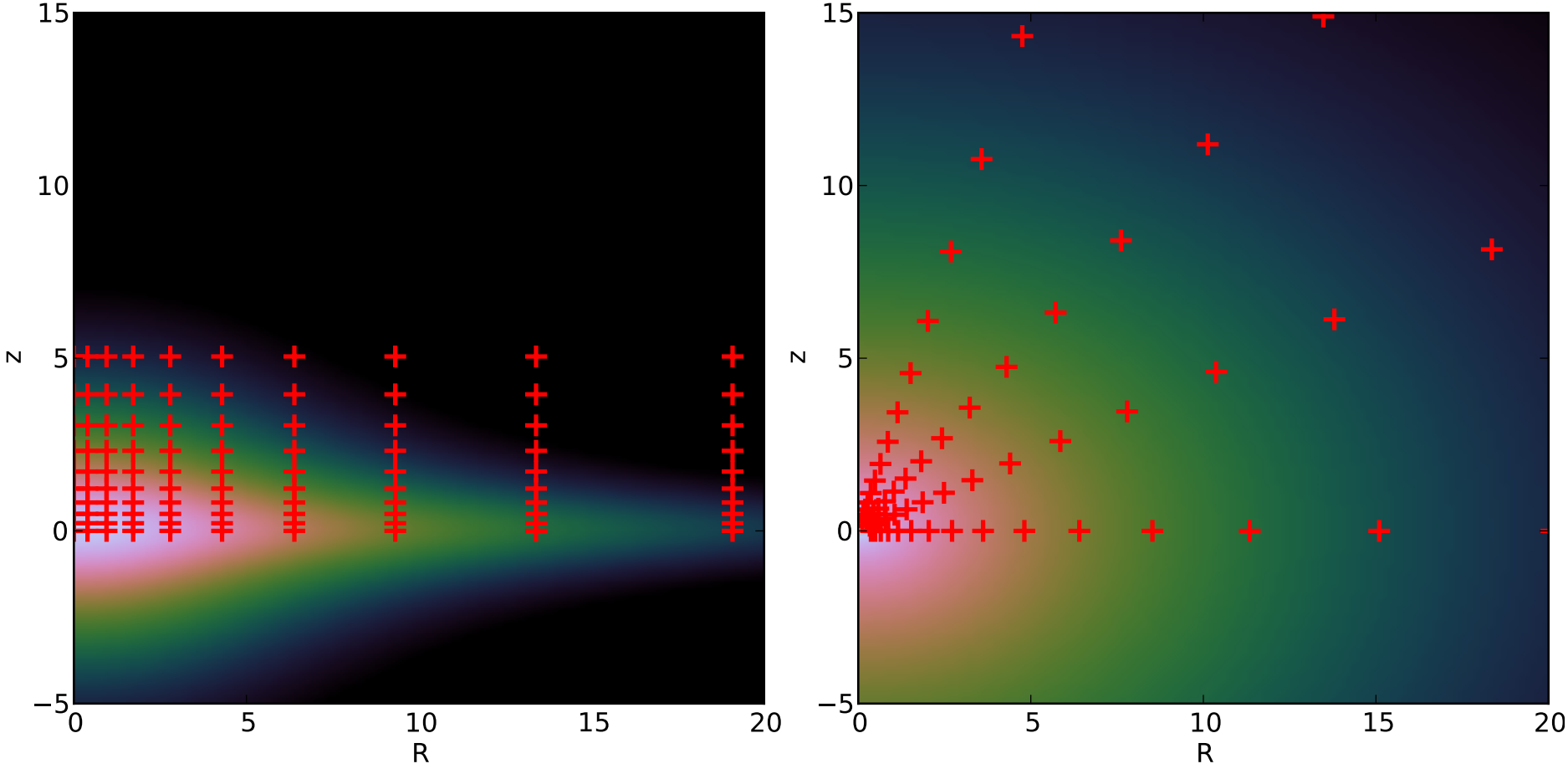
4. Basis-set expansion:

$$\Phi(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\Phi_{nlm}\,A_{nl}(r)\,Y_l^m(\theta,\phi).$$

(example: self-consistent field method of Hernquist&Ostriker 1992)

Two types of potential approximations used in models

- for disc-like components – azimuthal-harmonic expansion;
- for spheroidal components – spherical-harmonic expansion.



Gravitational potential extracted from N-body models

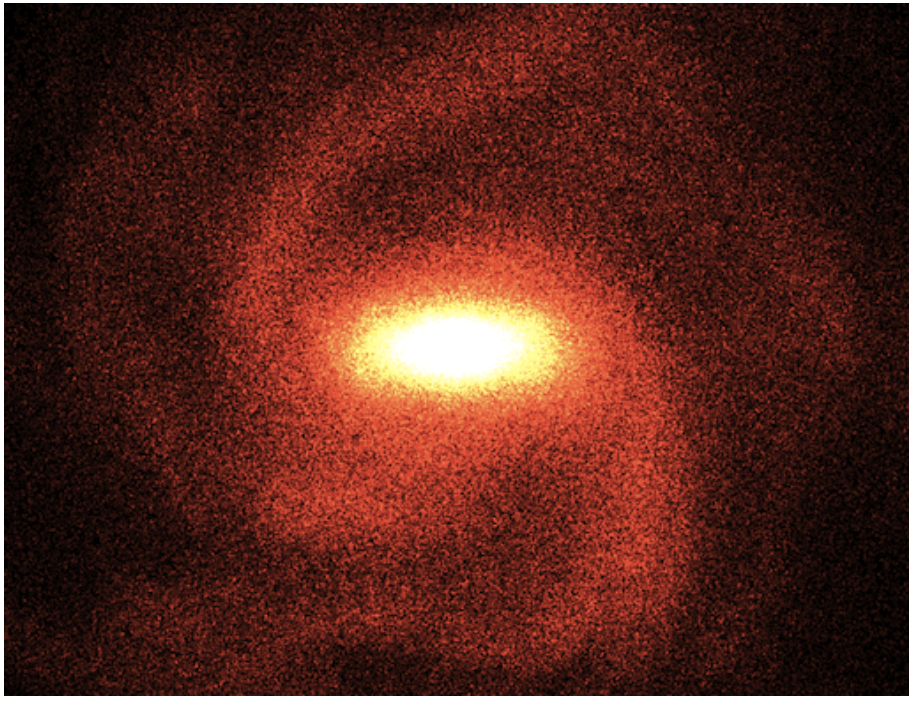
The spherical-harmonic and azimuthal-harmonic potential approximations can also be constructed from *N*-body models.

Advantages:

fast evaluation, smooth forces, suitable for orbit analysis.

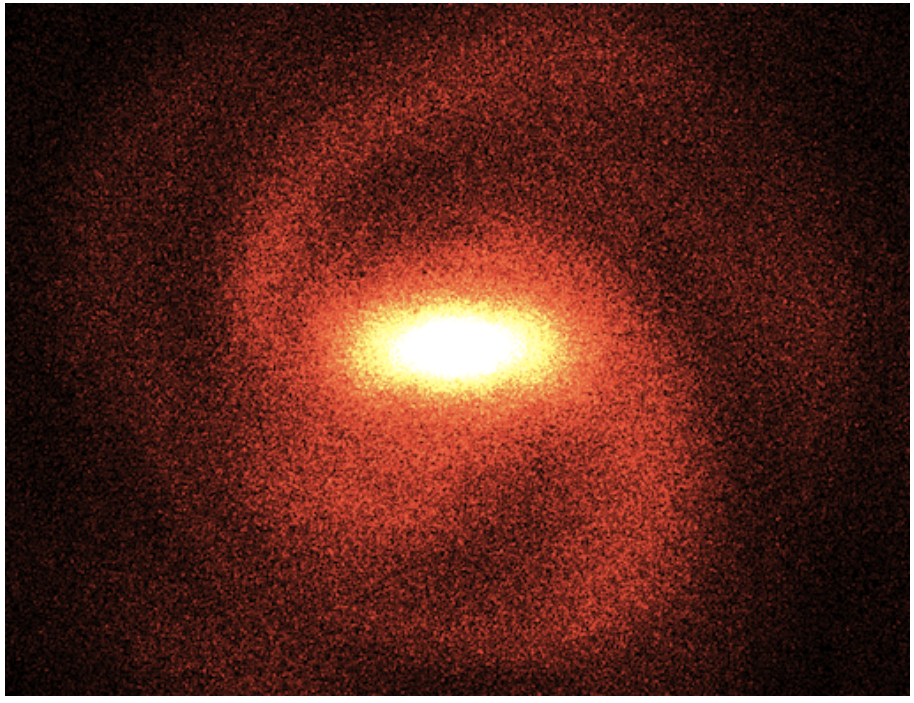
Real *N*-body model

(from Roca-Fabrega et al. 2013, 2014)



Potential approximation

(suitable for test-particle integrations, e.g. Romero-Gomez et al. 2011)



Models with prescribed density profile

Goal: $\rho(\mathbf{x}) \implies f(\mathcal{I})$

construct a self-consistent model for the given density profile (possibly with additional kinematic constraints).

Need to somehow invert the integral equation

$$\rho(\mathbf{x}) = \iiint d^3v \; f\big(\mathcal{I}(\mathbf{x},\mathbf{v})\big)$$

Variants of methods:

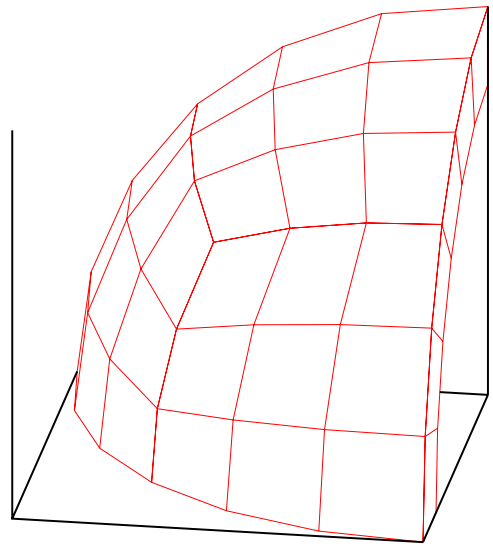
- Based on N-body models: made-to-measure and alike...
- Based on orbits: Schwarzschild's orbit-superposition method.
- Based on distribution-function "building blocks": this work.

Schwarzschild's orbit-superposition method

Discretize both the density profile and the distribution function:

$\rho(\mathbf{x}) \implies$ cells of a spatial grid;
mass of each cell is

$$m_c = \iiint_{\mathbf{x}\in V_c} \rho(\mathbf{x})\, d^3x$$



$f(\mathcal{I}) \implies$ collection of orbits:

$$f(\mathcal{I}) = \sum_{k=1}^{N_{\rm orb}} w_k\ \delta(\mathcal{I} - \mathcal{I}_k)$$

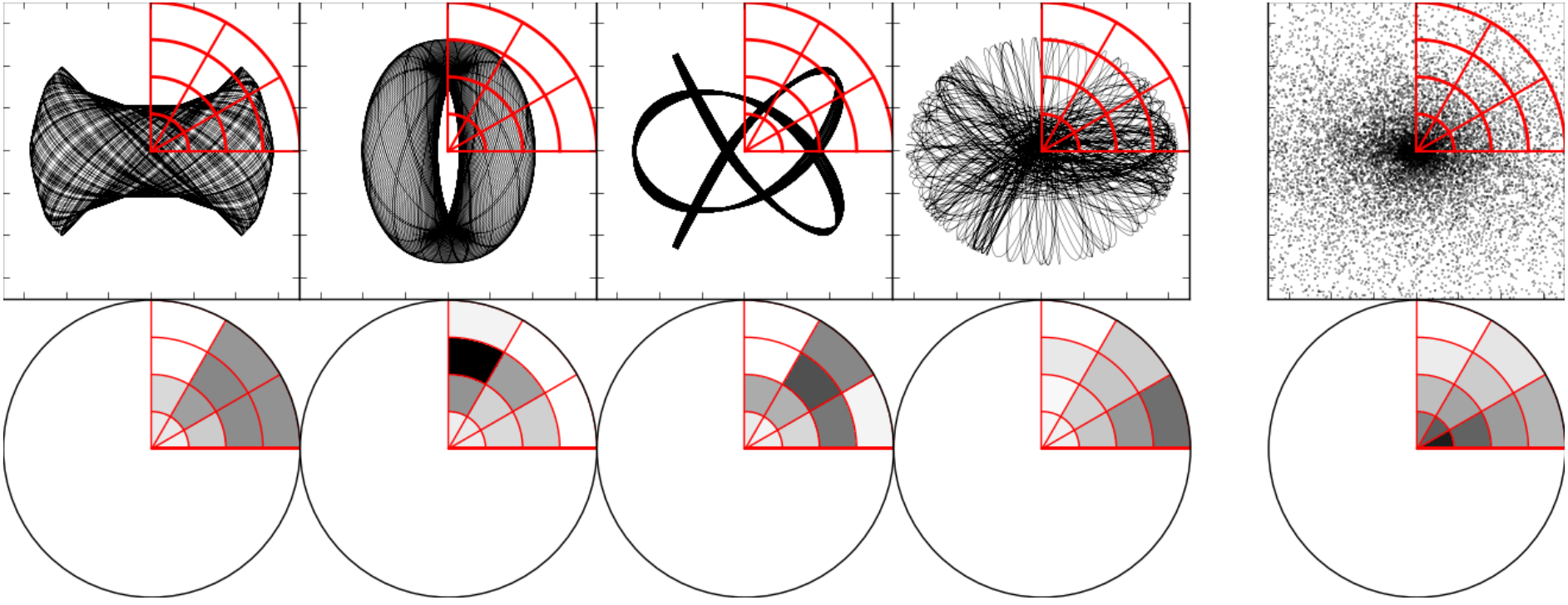
each orbit is a delta-function in the space of integrals

adjustable weight of each orbit

Schwarzschild's orbit-superposition method

orbits in the model

target density



discretized orbit density

(fraction of time t_{kc} that k-th orbit spends in c-th cell)

discretized density

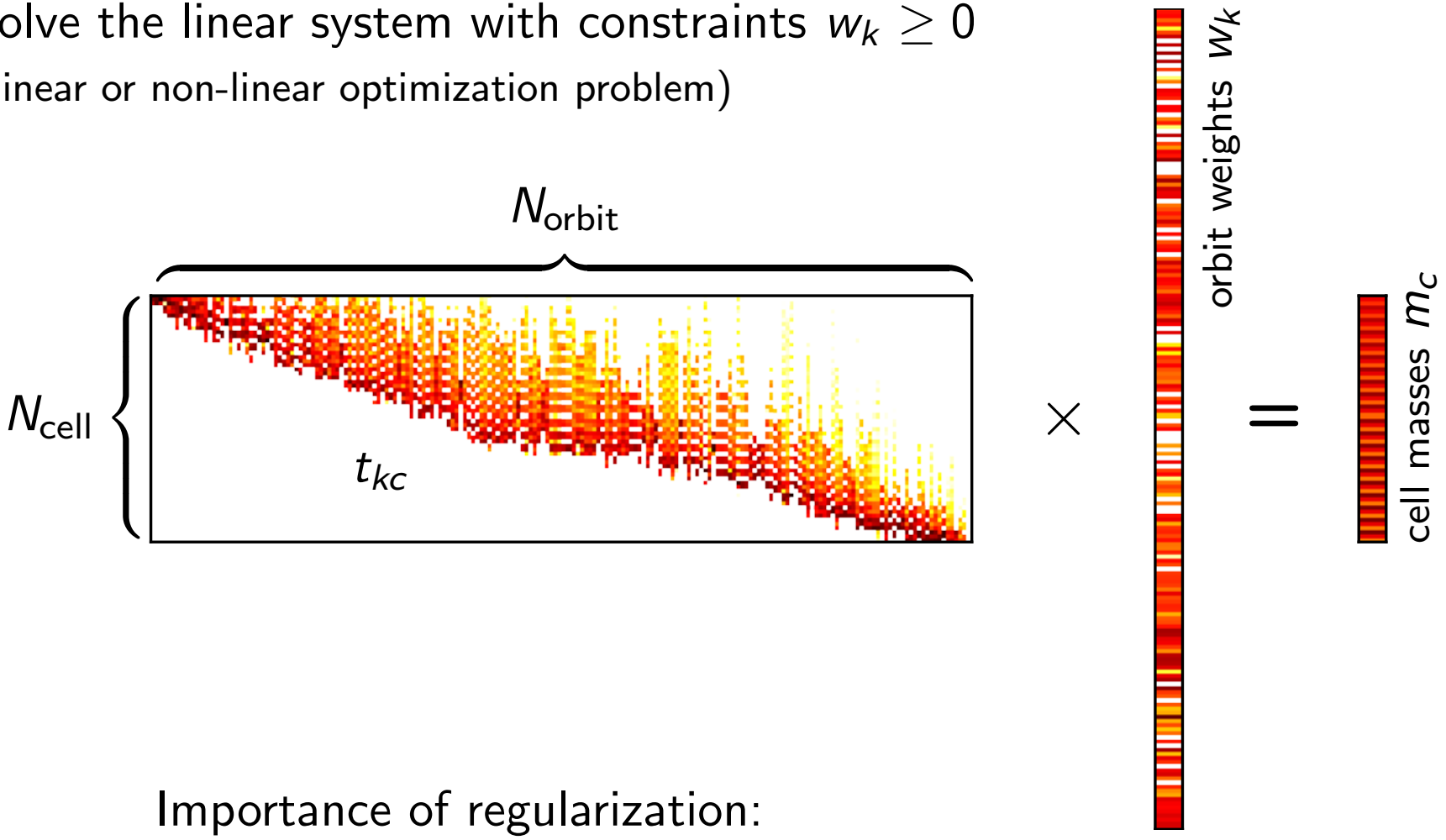
(mass m_c in grid cells)

For each c-th cell we require $\sum_k w_k\, t_{kc} = m_c$, where $w_k \geq 0$ is orbit weight

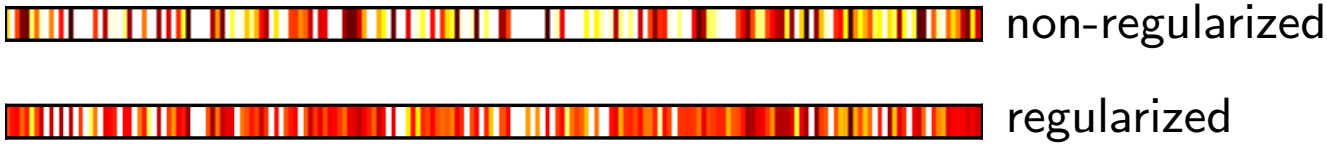
Schwarzschild's orbit-superposition method

Solve the linear system with constraints $w_k \geq 0$

(linear or non-linear optimization problem)



Importance of regularization:



Building blocks for distribution function

Same idea: invert the integral equation

$$\rho(\mathbf{x}) = \iiint d^3v \; f\big(\mathcal{I}(\mathbf{x},\mathbf{v})\big)$$

by decomposing f into a sum of building blocks (basis functions):

$$f(\mathcal{I}) = \sum_{k=1}^{N_{\text{basis}}} w_k \, f_k(\mathcal{I}),$$

computing the projections of all basis functions at a grid of points $\mathbf{x}_c$

$$\rho_{kc} \equiv \iiint d^3v \; f_k\big(\mathcal{I}(\mathbf{x}_c,\mathbf{v})\big),$$

and solving the optimization problem

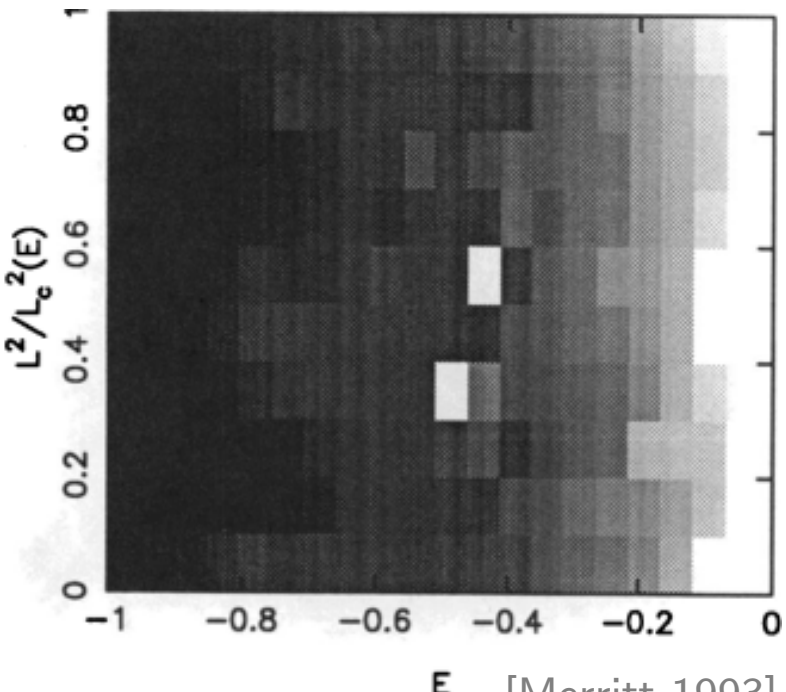
$$\rho_c \equiv \rho(\mathbf{x}_c) = \sum_{k=1}^{N_{\text{basis}}} w_k \, \rho_{kc} \,, \quad w_k \geq 0 \,, \quad c = 1..N_{\text{constraint}},$$

to find the weights of basis functions w_k.

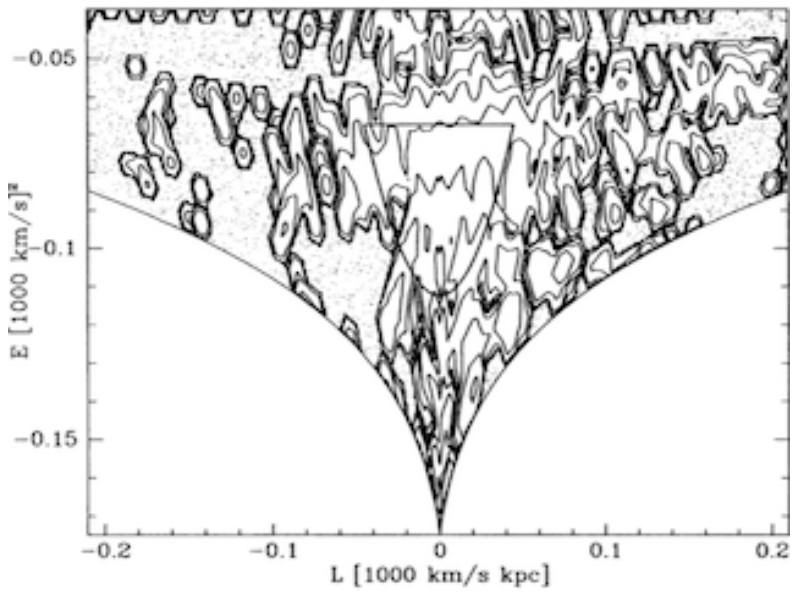
Building blocks for distribution function

Similar approaches suggested previously:

- Dejonghe(1989), Merritt&Saha(1993): $f_k(E, L)$ as Fricke components $|E|^{\alpha}\, L^{-2\beta}$;
- Merritt(1993,1996): histograms (Π-shaped blocks) for $f(E, L)$ or $f(E, L_z)$;
- Kuijken(1994), Pichon&Thiébaut(1998): bilinear interpolation for $f(E, L_z)$;
- Dehnen&Gerhard(1994): Chebyshev polynomial basis for $f(E, L_z)$;
- Magorrian(2014): superposition of multivariate Gaussian 'blobs' for $f(E, L)$.

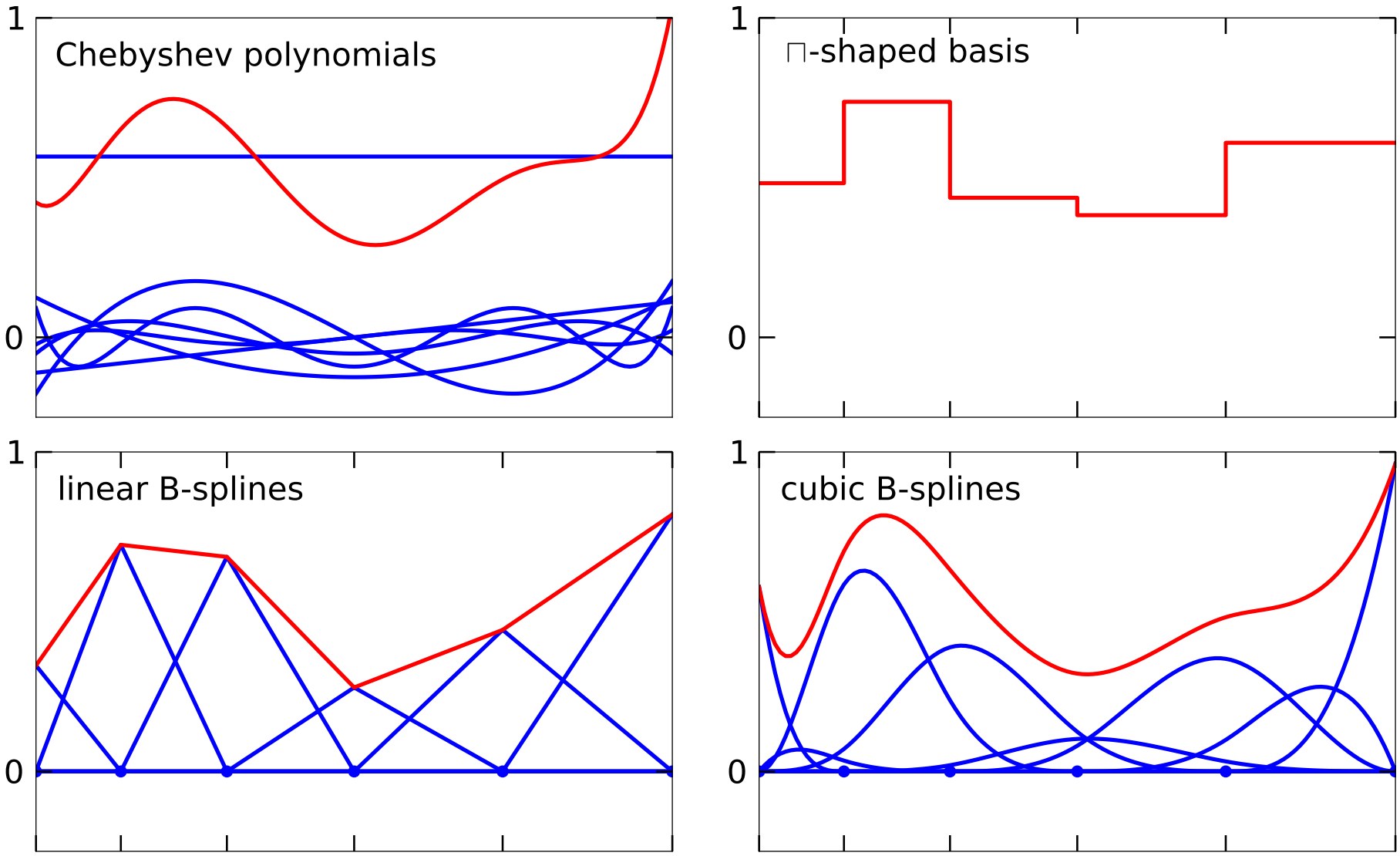


[Merritt 1993]



[Kuijken 1994]

Building blocks for interpolation



B-splines of degree N:

flexible choice of grid points, locality, smoothness (increases with N), nonnegativity.

Models with non-parametric distribution function

- $f(\mathbf{J})$ represented as an interpolated 3d function in action space (tensor product of 1d B-splines);
- weights of basis functions found by solving a linear or quadratic optimization problem (constraints: values of density at a 3d grid in space);
- smoothness: choice of degree of B-splines;
- regularization: minimum-curvature condition for the 3d interpolant (roughness penalty);
- possibility of determining $f(\mathbf{J})$ from an N-body model or from discrete observational points, using maximum penalized likelihood method (work in progress..).

Advantages of models based on distribution function

- Clear physical meaning
  (localized structures in the space of integrals of motion);
- Easy to compare different models
  (how to compare two N-body or N-orbit models?);
- Easy to compare models to discrete observational data;
- Easy to sample particles from the distribution function
  (convert to an N-body model);
- Starting point for stability analysis and kinetic theory
  (perturbation theory and relaxation most naturally formulated in terms of actions and frequencies);

Caveats:

- Implicitly rely on the integrability of the potential,
  ignore the presence of resonant orbit families;
- So far implemented only for axisymmetric models
  (not a fundamental limitation).

AGAMA library – Action-based galaxy modeling architecture

- Extensive collection of gravitational potential models
  (analytic profiles, azimuthal- and spherical-harmonic expansions);
- Conversion to/from action/angle variables
  (fast and accurate method for spherical potentials, Stäckel fudge for axisymmetric potentials, torus mapping);
- Action-based distribution functions; generation of N-body models and determination of best-fit parameters of DF and potential;
- Self-consistent multicomponent models with action-based DFs:
  (iterative method for $f(\mathbf{J}) \Rightarrow \rho(\mathbf{x})$, non-parametric DF recovery $\rho(\mathbf{x}) \Rightarrow f(\mathbf{J})$);
- Efficient and carefully designed C++ implementation, examples, Python interface, compatibility with other software such as galpy;

`https://github.com/GalacticDynamics-Oxford/Agama`

Conclusions

- Advantage of models based on distribution functions;
- Advantage of actions as arguments of distribution functions;
- Two approaches for construction of self-consistent models;
- Software available for the community.

THANK YOU!