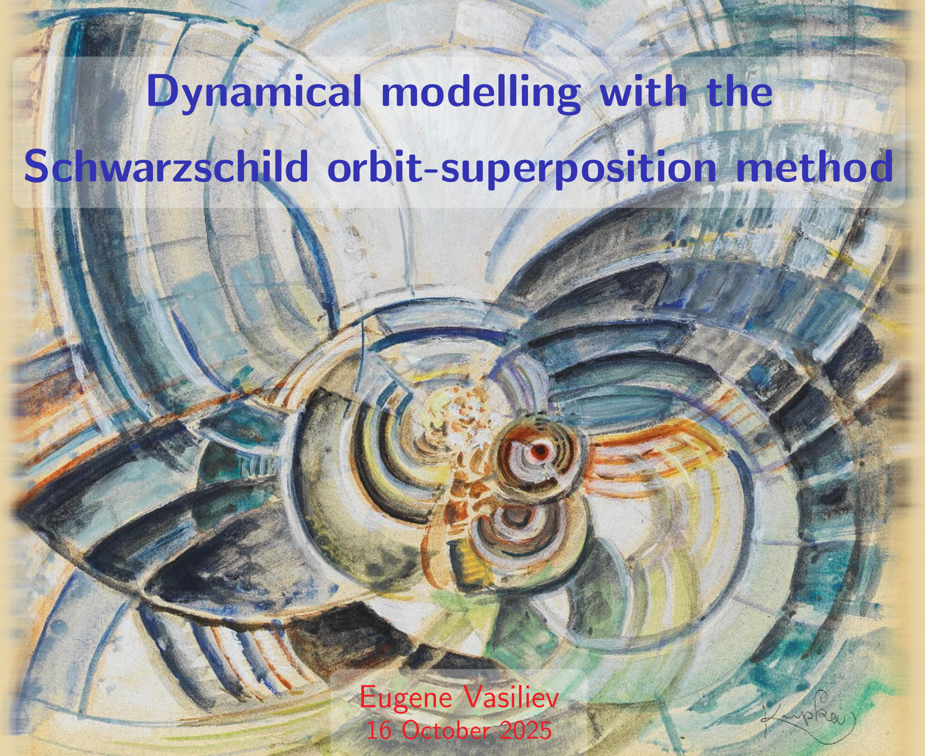
Dynamical modelling with the Schwarzschild orbit-superposition method



Eugene Vasiliev

16 October 2025

Self-consistent equilibrium models of stellar systems

Definition: a stellar system described by a time-independent DF $f\big(\mathcal{I}(\mathbf{x},\mathbf{v};\ \Phi)\big)$ and potential $\Phi(\mathbf{x})$, which are related by the Poisson equation:

integrals of motion (Jeans thm)

$$\nabla^2\Phi(\mathbf{x}) = 4\pi\, G\, \rho(\mathbf{x}), \quad \text{where}\ \ \rho(\mathbf{x}) = \iiint \mathrm{d}^3\mathbf{v}\ f\big(\mathcal{I}(\mathbf{x},\mathbf{v})\big).$$

Applications:

- inference on gravitational potential from stellar kinematics (so-called *dynamical modelling*)
- creation of initial conditions for isolated galaxy simulations

Methods: (non-exhaustive list)

- Jeans modelling
- Distribution function-based approaches
- Schwarzschild orbit-superposition method
- Guided *N*-body simulations (made-to-measure)

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- Schwarzschild orbit-superposition method
- Guided *N*-body simulations (made-to-measure)

Monica Valluri (U.Michigan)

Orbit-superposition method for self-consistent models

Introduced by Schwarzschild (1979) as a practical approach
for constructing self-consistent triaxial models with prescribed $\rho(\mathbf{x}) \Leftrightarrow \Phi(\mathbf{x})$.

integrals of motion

To invert the equation $\rho(\mathbf{x}) = \iiint f\left(\mathcal{I}\left(\mathbf{x}, \mathbf{v}\,;\ \Phi\right)\right)\, d^3\mathbf{v}$,

discretize both the density profile and the distribution function:

$\rho(\mathbf{x}) \implies$ cells of a spatial grid;
mass of each cell is $M_c = \iiint\limits_{\mathbf{x}\in V_c} \rho(\mathbf{x})\, d^3x$;

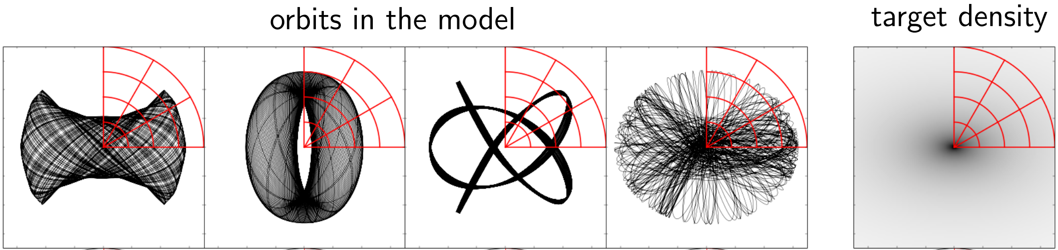
$f(\mathcal{I}) \implies$ collection of orbits with unknown weights:

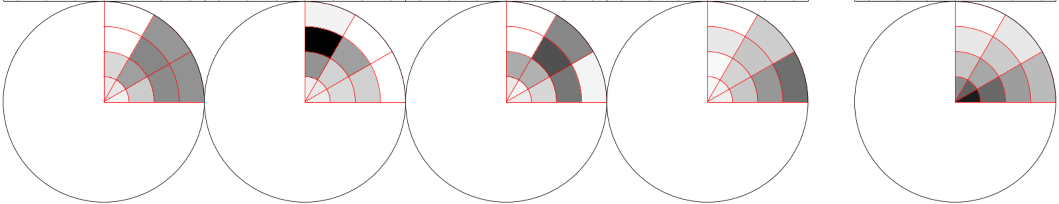
$$f(\mathcal{I}) = \sum_{k=1}^{N_{\rm orb}} w_k\ \delta(\mathcal{I} - \mathcal{I}_k)$$

each orbit is a delta-function in the space of integrals of motion

adjustable weight of each orbit [to be determined]

Schwarzschild's orbit-superposition method: self-consistency





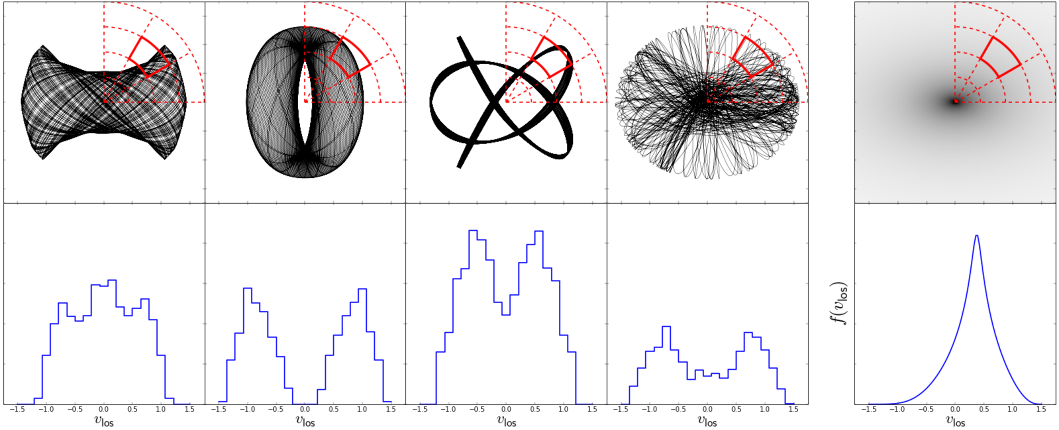
discretized orbit density
(fraction of time t_{kc} that k-th orbit spends in c-th cell)

discretized density
(mass M_c in grid cells)

For each c-th cell we require $\sum_k w_k\, t_{kc} = M_c$, where $w_k \geq 0$ is orbit weight

Schwarzschild's orbit-superposition method: kinematics

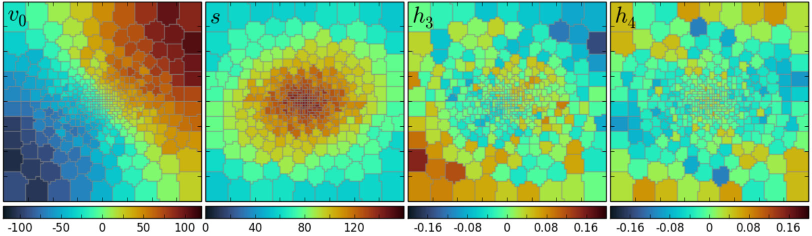
orbits in the model

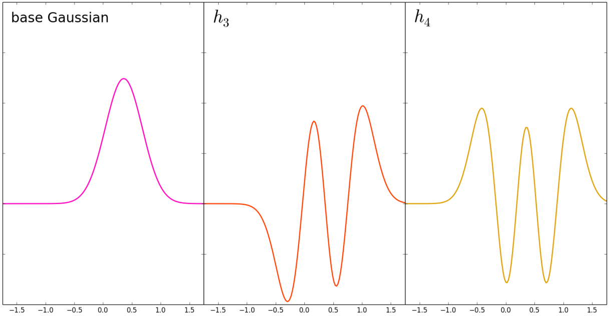


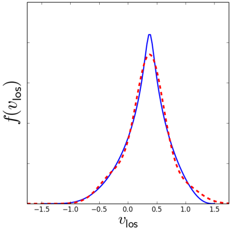
orbit LOSVDs

target LOSVD

Schwarzschild's orbit-superposition method: kinematics



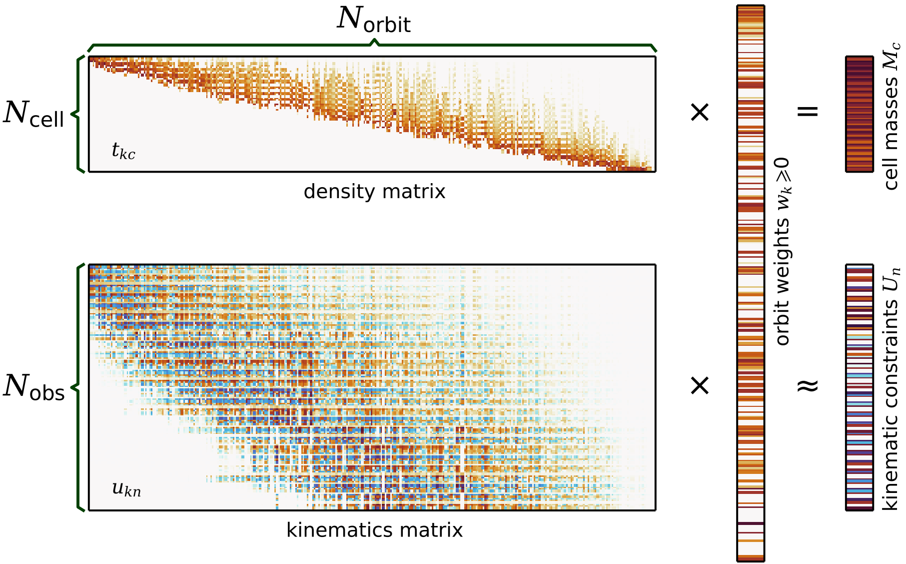




Gauss–Hermite parametrization of LOSVDs [van der Marel & Franx 1993; Gerhard 1993]

Schwarzschild's orbit-superposition method: fitting procedure

Solve the linear system with non-negativity constraints on the solution vector $w_k \geq 0$ (linear or non-linear optimization problem)



Schwarzschild's orbit-superposition method: fitting procedure

- Assume some potential $\Phi(\mathbf{x})$
  (e.g., from the deprojected luminosity profile plus parametric DM halo or SMBH)
- Construct the orbit library in this potential:
  for each k-th orbit, store its contribution to the discretized density profile
  t_{kc}, $c = 1..N_\mathrm{cell}$ and to the kinematic observables u_{kn}, $n = 1..N_\mathrm{obs}$
- Solve the constrained optimization problem to find orbit weights w_k:

$$\text{minimize } \chi^2 + \mathcal{S} \equiv \sum_{n=1}^{N_\mathrm{obs}} \left(\frac{\sum_{k=1}^{N_\mathrm{orb}} w_k\, u_{kn} - \overset{\text{observational constraints}}{U_n}}{\underset{\text{their uncertainties}}{\delta U_n}} \right)^2 + \underset{\text{regularization term}}{\mathcal{S}}(\{w_k\})$$

$$\text{subject to } w_k \ge 0, \quad k = 1..N_\mathrm{orb},$$

$$\sum_{k=1}^{N_\mathrm{orb}} w_k\, t_{kc} = \underset{\text{density constraints (cell masses)}}{M_c}, \quad c = 1..N_\mathrm{cell}$$

- Repeat for different choices of potential and find the one that has lowest χ^2

github.com/GalacticDynamics-Oxford/Agama/blob/master/py/tutorial_schwarzschild.ipynb

Schwarzschild orbit-superposition modelling

This approach for constructing equilibrium models was introduced by Martin Schwarzschild in 1979 (after his retirement!), as the first practical demonstration that triaxial equilibrium galaxy models may exist – at the time, it was not at all obvious!

Originally, the method was purely a theorist's tool, but soon it was extended to take into account observational constraints and became one of standard tools in dynamical modelling (inferring the mass distribution from kinematic data). At the same time, its theoretical applications include the construction of initial conditions for N-body simulations. In both theoretical and observational flavours, its advantages are the ability to handle complicated geometry of the model (axisymmetric, triaxial and even a rotating bar) and a very flexible representation of the distribution function – the latter can also be a drawback, in the sense that the resulting models may have unphysically rapid variations of the DF across the phase space. To mitigate this situation, some regularisation is needed, as with any non-parametric method.

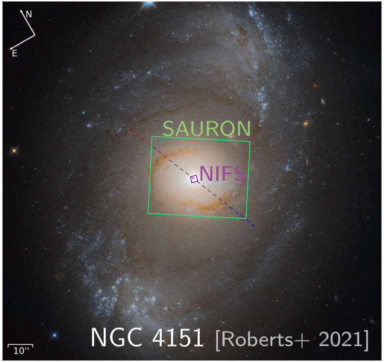
In this tutorial, we first discuss the fundamental concepts of the orbit-superposition method and the particular features of its implementation in `Agama` – essentially, the "theoretical" flavour of the method. Observational applications will be considered in the second part of this tutorial. Since the method works with orbits, it is essential to first complete the `tutorial_potential_orbits` notebook.

Observational applications of the Schwarzschild method

Photometry: usually HST (FoV $\sim 3'$, PSF $\gtrsim 0.05''$)

Kinematics: integral-field spectroscopic units (IFU):

- ground-based, non-AO: SAURON (WHT 4m, FoV $\sim 35''$, PSF $\sim$ 1–2$''$), MUSE (VLT 8m, FoV $60''$, PSF $\sim 1''$)
- ground-based, AO: NIFS (Gemini 8m), SINFONI (VLT 8m): FoV $\sim 3''$, PSF $\sim 0.1''$
- space-based: NIRSpec (JWST 6.5m): FoV $\sim 3''$, PSF $\sim 0.1''$ (but higher contrast)



NGC 4151 [Roberts+ 2021]

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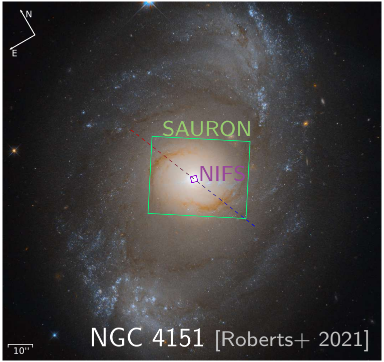


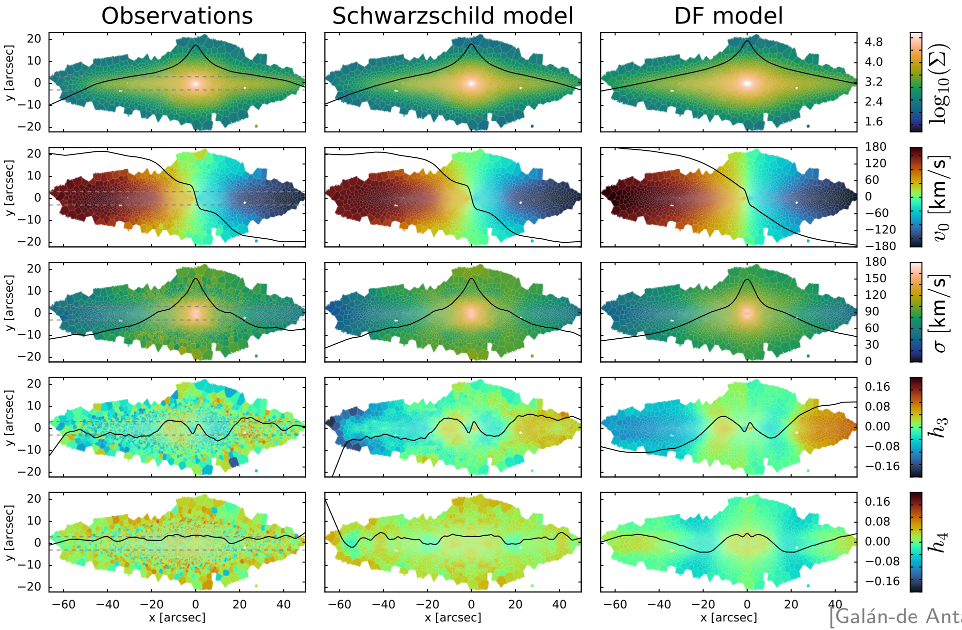
NGC 4151 [Roberts+ 2021]

Modelling code

[Vasiliev & Valluri 2020]

Theme 1: DF- and orbit-based dynamical models

Model of an edge-on S0 galaxy FCC 170 constrained by MUSE IFU kinematics

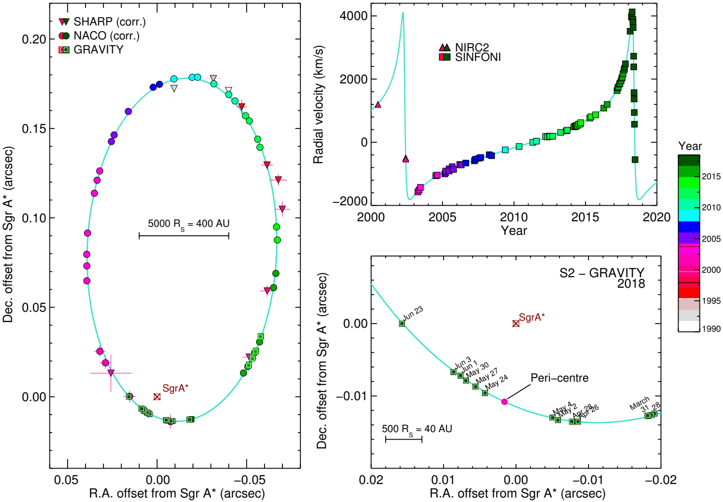


[Galán-de Anta+ 2023]

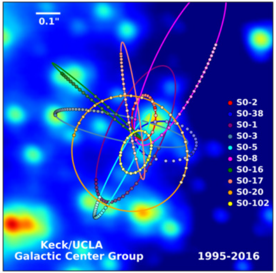
Theme 2: measurement of supermassive black hole masses

Theme 2: measurement of supermassive black hole masses

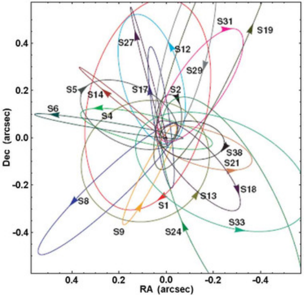
- time-resolved stellar orbits around Sgr A*



[VLT/GRAVITY]



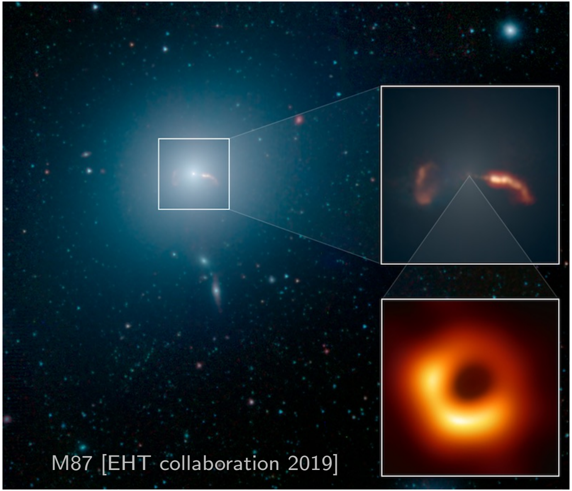
[Keck: Ghez et al.]



[VLT: Genzel et al.]

Theme 2: measurement of supermassive black hole masses

- time-resolved stellar orbits around Sgr A*
- imaging of the inner edge of the accretion disc by the Event Horizon telescope



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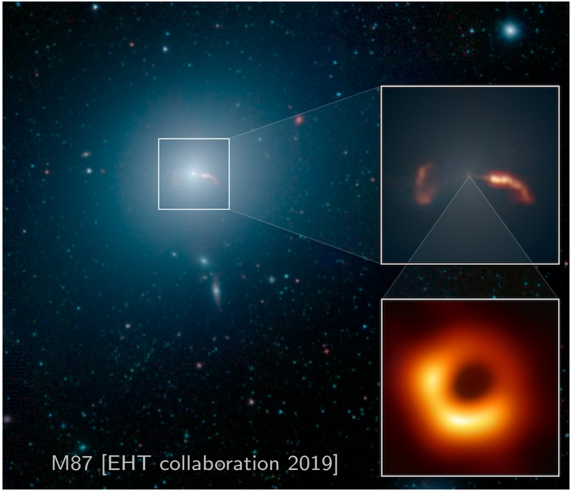
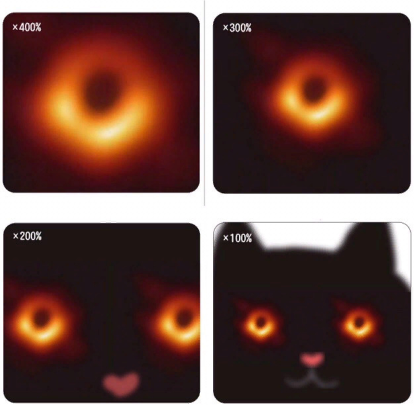


(in popular culture)



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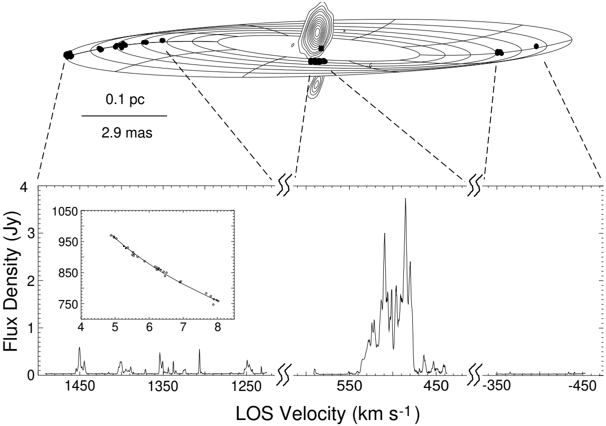
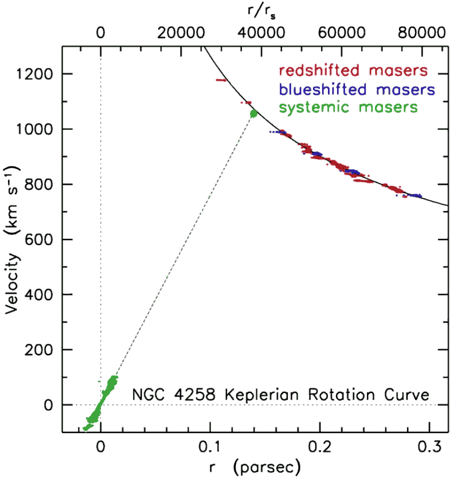
- radiointerferometry of megamasers in the accretion disc





NGC 4258 [Herrnstein+ 2005]

Theme 2: measurement of supermassive black hole masses

- time-resolved stellar orbits around Sgr A*
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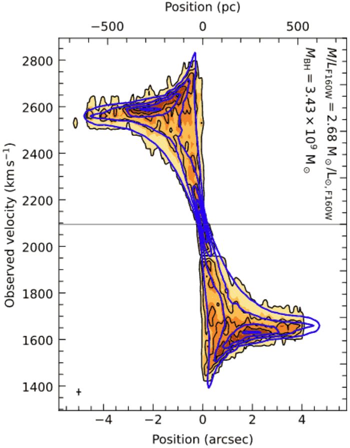
- kinematics of molecular gas in the accretion disc with ALMA interferometry



NGC 4751 [Dominiak+ 2024]

Theme 2: measurement of supermassive black hole masses

- time-resolved stellar orbits around Sgr A*
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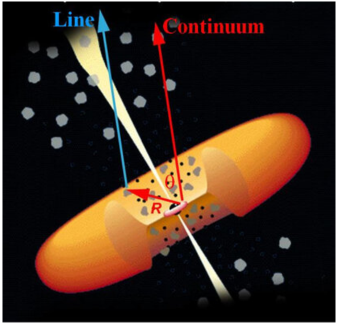
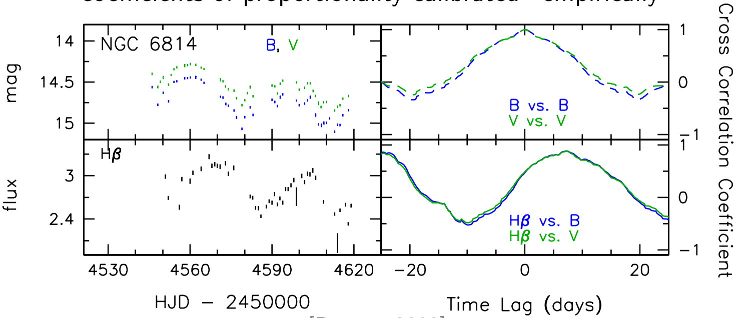
- reverberation mapping (time-delay variability monitoring in AGN)

$$GM_\bullet \propto R_{\mathrm{BLR}}\,\sigma^2, \qquad R_{\mathrm{BLR}} \propto c\,\Delta t$$

coefficients of proportionality calibrated "empirically"



[Kaspi 2018]



[Bentz+ 2009]

Theme 2: measurement of supermassive black hole masses

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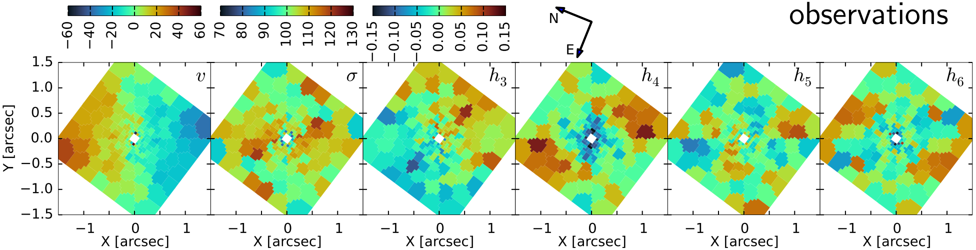
- kinematics of molecular gas in the accretion disc with ALMA interferometry
- reverberation mapping (time-delay variability monitoring in AGN)
- kinematics and dynamical modelling of unresolved stars in galactic nuclei (Jeans and Schwarzschild models)

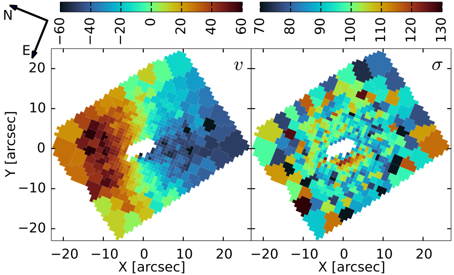
Theme 2a: stellar-dynamical measurements of SMBH in AGN

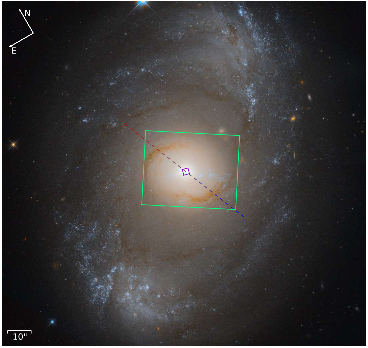
observations

NIFS



SAURON

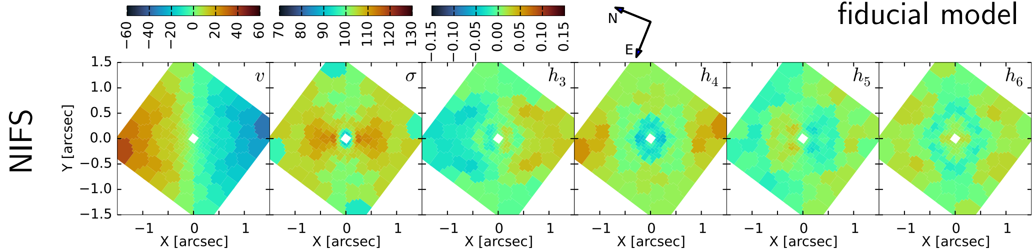


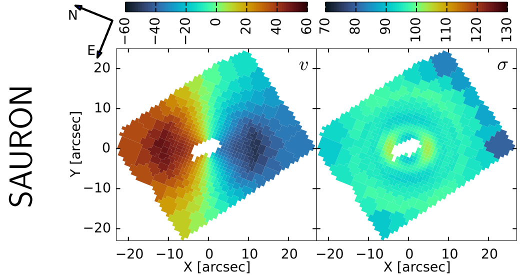


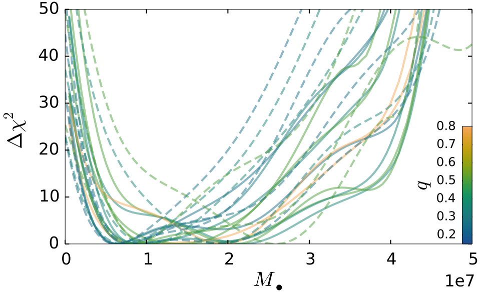
NGC 4151 [Roberts+ 2021]

Theme 2a: stellar-dynamical measurements of SMBH in AGN

fiducial model





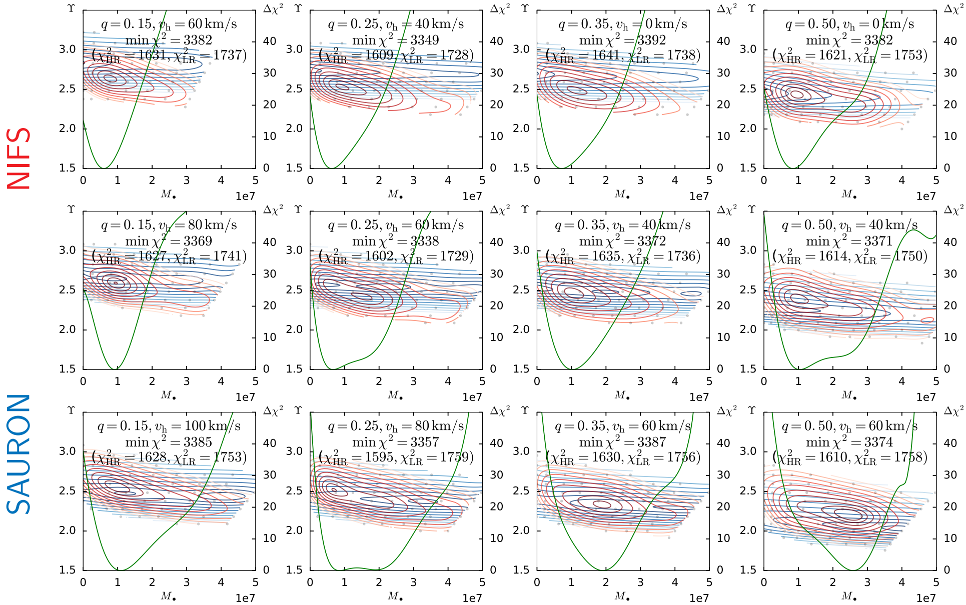


NGC 4151 [Roberts+ 2021]

SD: $M_\bullet = (0.25 \div 3.0) \times 10^7\, M_\odot$

RM: $M_\bullet = 2 \times 10^7\, M_\odot$

Theme 2a: stellar-dynamical measurements of SMBH in AGN

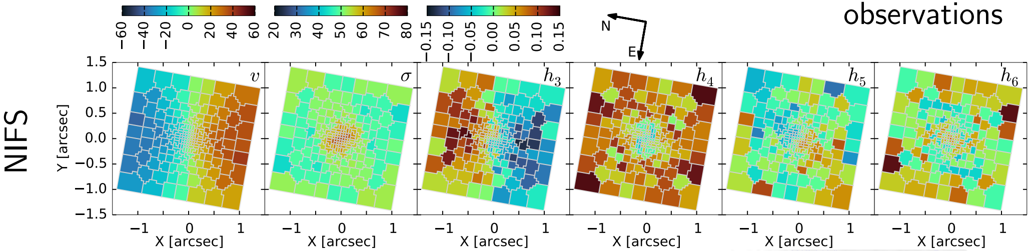


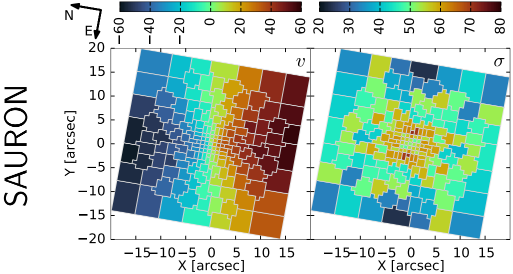
NGC 4151 [Roberts+ 2021]

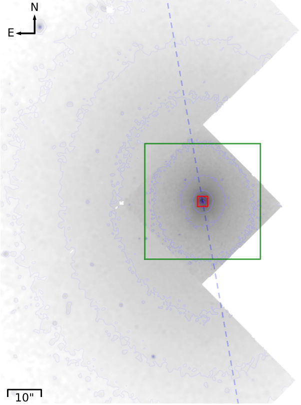
variants of models

Theme 2a: stellar-dynamical measurements of SMBH in AGN

observations

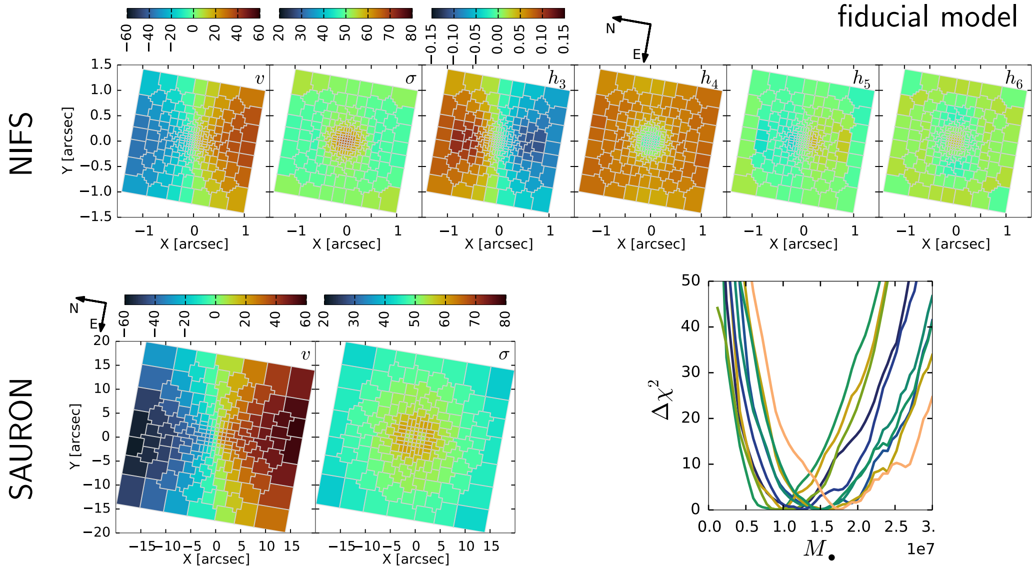






NGC 5273 [Merrell+ 2023]

Theme 2a: stellar-dynamical measurements of SMBH in AGN

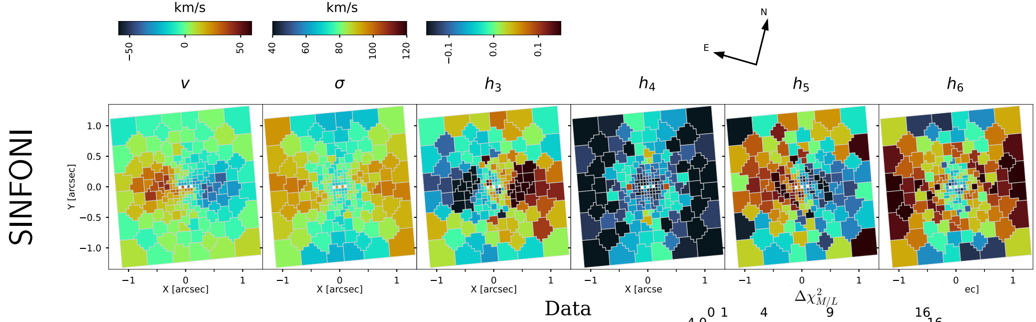


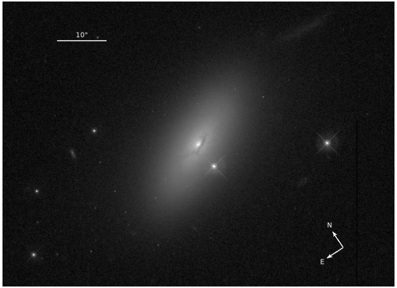
NGC 5273 [Merrell+ 2023]

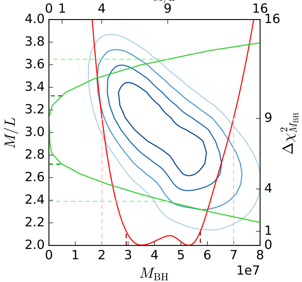
SD: $M_\bullet = (0.5 \div 2.0) \times 10^7\ M_\odot$

RM: $M_\bullet = (2.1 \div 6.3) \times 10^7\ M_\odot$

Theme 2a: stellar-dynamical measurements of SMBH in AGN





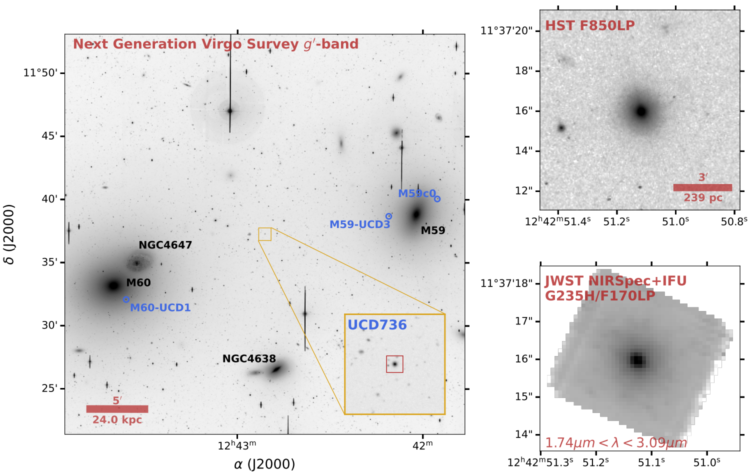


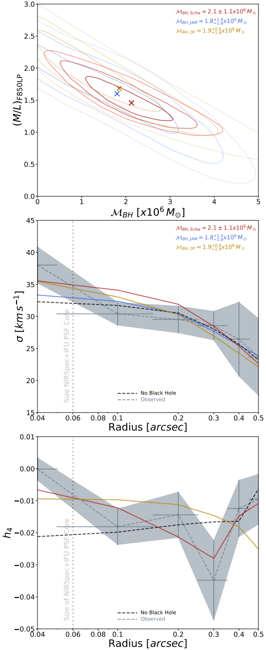
MCG-06-30-15 [Das+ 2025]

SD: $M_\bullet = (3.0 \div 5.8) \times 10^7\ M_\odot$

RM: $M_\bullet = (0.7 \div 7.6) \times 10^6\ M_\odot$

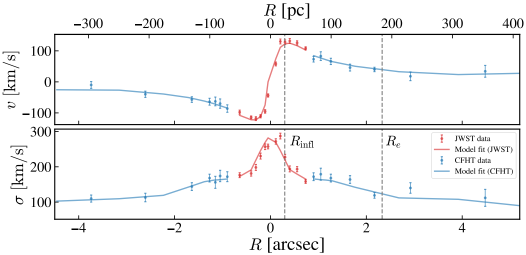
Theme 2b: SMBH in ultracompact galaxies with JWST

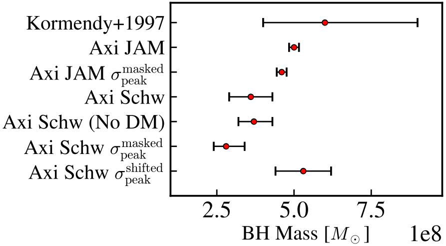


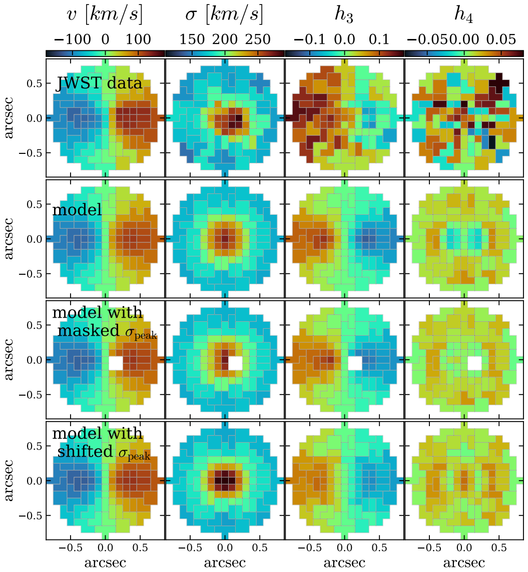


UCD736 [Taylor+ 2025]: $M_\bullet = (2.0 \pm 1) \times 10^6\, M_\odot$
using Jeans, DF and Schwarzschild methods (spherical)
(overmassive SMBH, $M_\bullet \sim 0.1\, M_\star$)

Theme 2b: SMBH in ultracompact galaxies with JWST







NGC 4486b [Tahmasebzadeh+ 2025]:

$M_\bullet = (3.6 \pm 0.7) \times 10^8\, M_\odot$

(overmassive off-centred SMBH, $M_\bullet \sim 0.1\, M_\star$)

Theme 3: modelling of barred galaxies

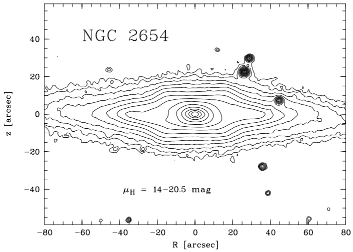
Challenges: triaxial geometry, chaotic regions in phase space

Goals:	Ω	Φ
Jeans modelling	$-$	$-$
Distribution functions, e.g., $f(\mathbf{J})$	?	?
Tremaine–Weinberg	$\pm$	$-$
Orbital response models	$+$	$+$
Guided N-body simulations (made-to-measure)	$+$	$+$
Schwarzschild orbit-superposition modelling	$+$	$+$

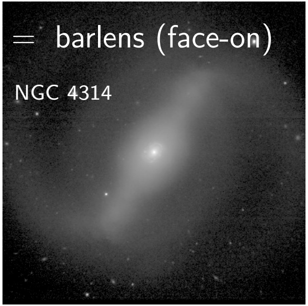
3D structure of bars

Bars often buckle vertically from the disk plane, but only in the inner part where the planar orbits are unstable;
shorter and vertically thick part is associated with boxy/peanut (B/P) bulges, and the longer and thinner component can be seen in face-on barlens galaxies [Athanassoula 2005, 2013].

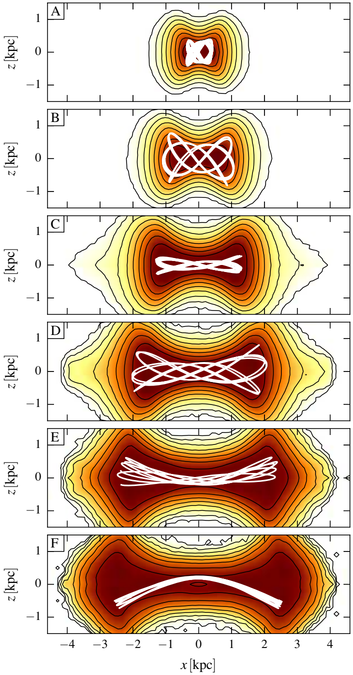
boxy/peanut bar (edge-on) = barlens (face-on)



[Lütticke+ 2000]



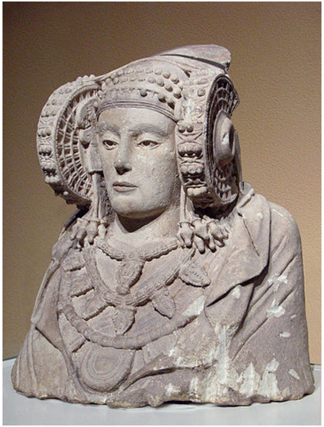
[Laurikainen+ 2011]



[Portail+ 2015]

The deprojection problem

$\Longrightarrow$



Pathway from 2d surface brightness profile to 3d density profile is non-unique

(presented at the bars conference in Granada, July 2023)

The deprojection problem

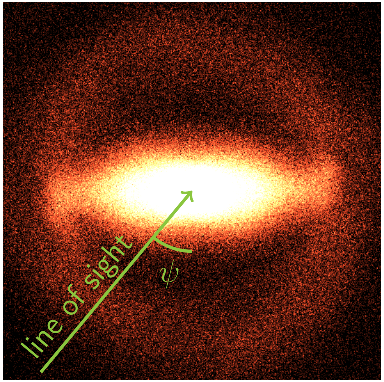
$\Longrightarrow$

Pathway from 2d surface brightness profile to 3d density profile is non-unique

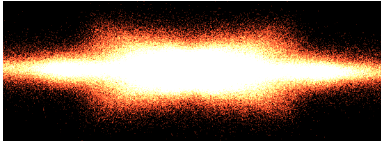
(talk at the Academy of Athens, May 2024)

Forward-modelling of 3d density profile

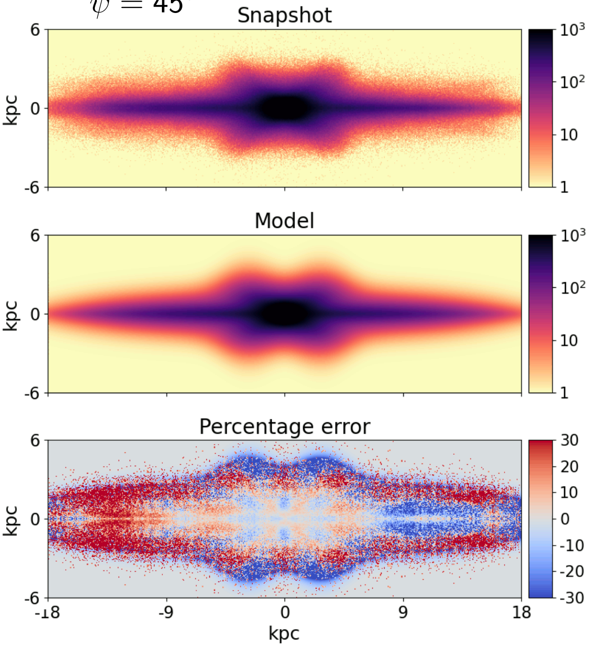
face-on view



edge-on view, $\psi = 0$



$\psi = 45^\circ$

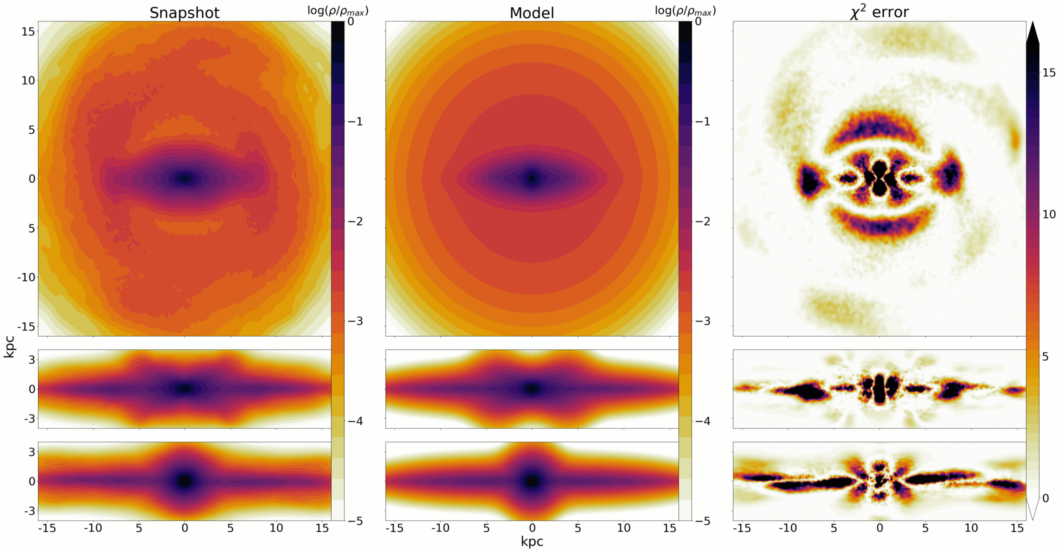


[Dattathri+ 2024]

Forward-modelling of 3d density profile

The fitted model qualitatively recovers the 3d density profile, though not without some defects

$\psi = 45^\circ$



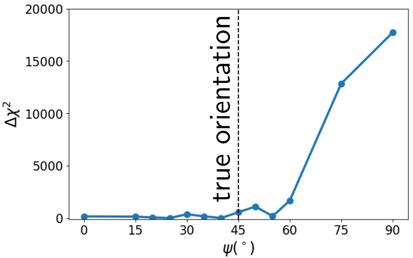
[Dattathri+ 2024]

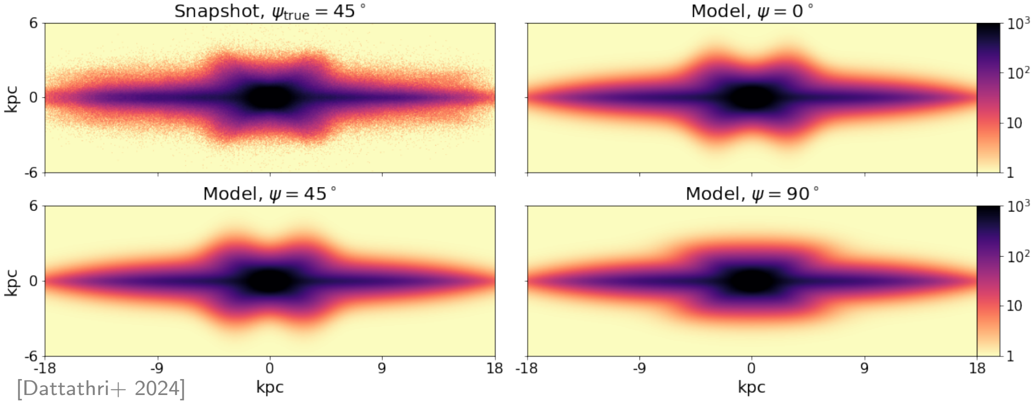
Degeneracies in determining bar orientation

It is impossible to distinguish a rotated bar ($0 < \psi < i_{max} \lesssim 90^\circ$) from a shorter bar viewed at $\psi = 0^\circ$ just from photometry.

(It might be easier at lower inclinations $i < 90^\circ$).

Kinematics / dynamical modelling should help?

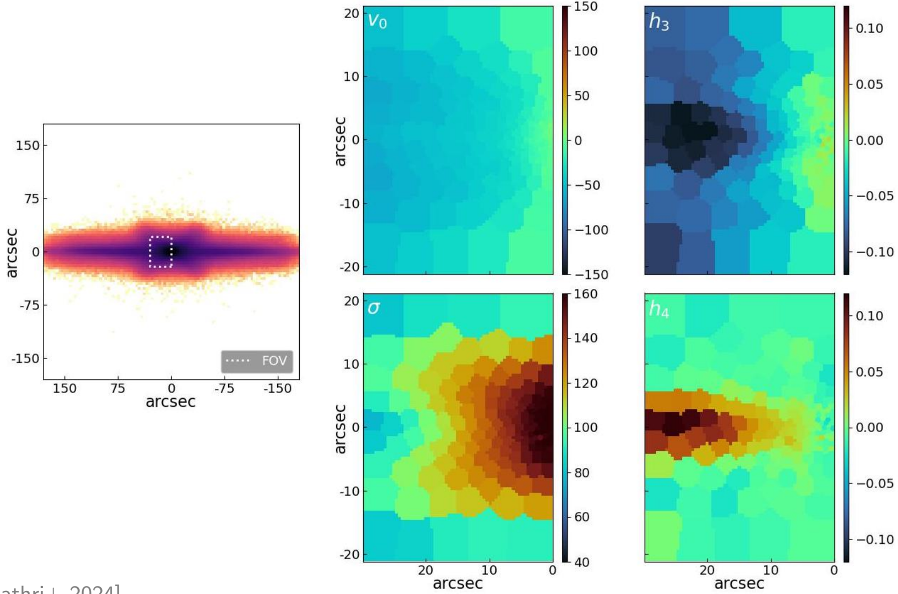




[Dattathri+ 2024]

Schwarzchild modelling of deprojected bars

MUSE-like kinematic maps ($1'$ FoV) of a Milky Way-like galaxy at $D = 20$ Mpc

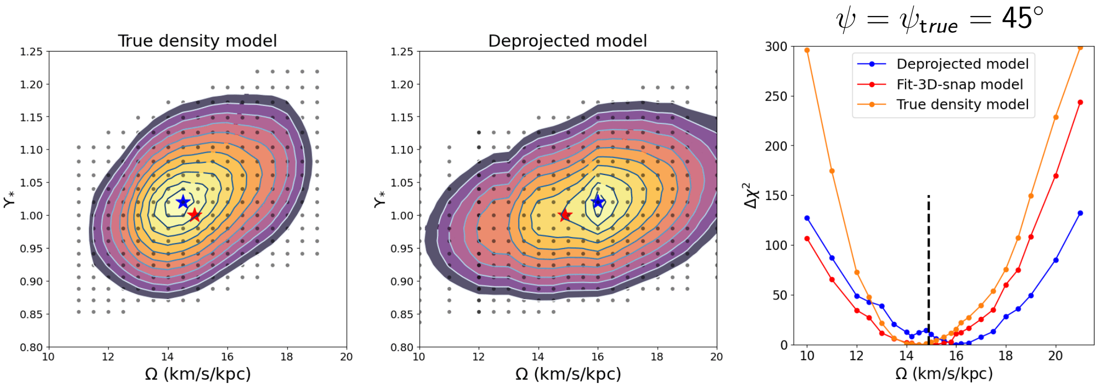


[Dattathri+ 2024]

Recovery of bar pattern speed

Ω is recovered almost perfectly if the true 3d density is used, or to within 10% if the deprojected density is used.

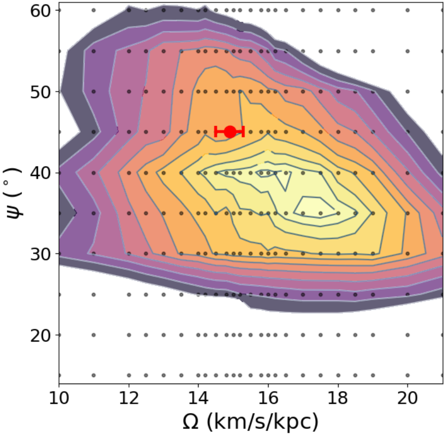
This is for the most challenging edge-on orientation, where the Tremaine–Weinberg method is not applicable!

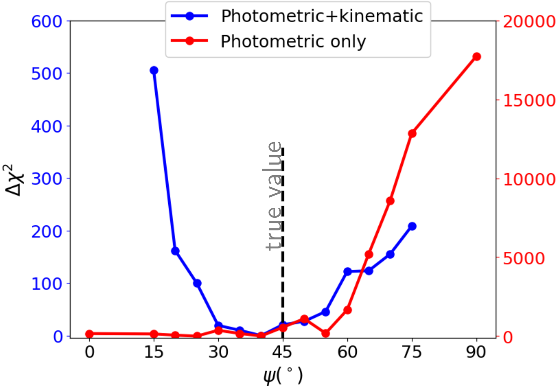


[Dattathri+ 2024]

Recovery of bar orientation

Bar orientation is also constrained much better than from pure photometry



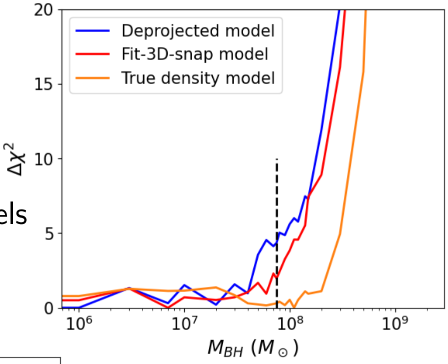


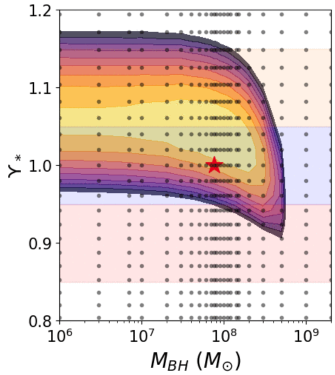
[Dattathri+ 2024]

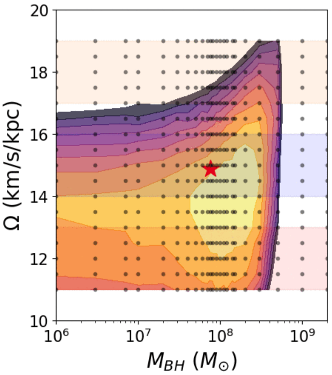
Recovery of black hole mass

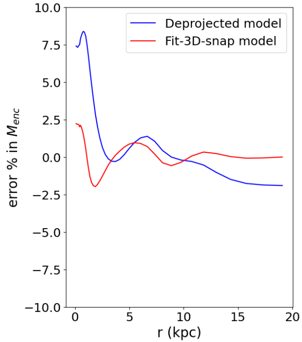
Central supermassive black hole

- does not destroy the bar [Wheeler+ 2023]
- has only an upper limit on $M_\bullet$ in these models
- is very sensitive to the accuracy of reconstruction of enclosed stellar mass









Theme 4: Milky Way halo

to be continued...