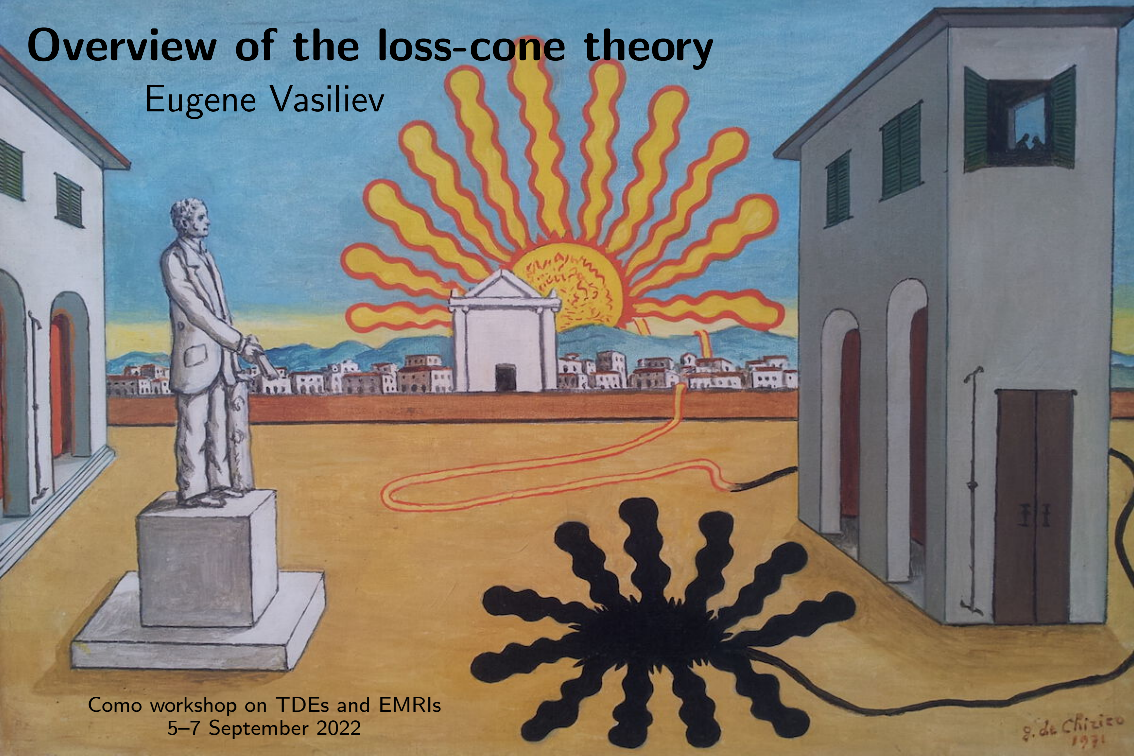
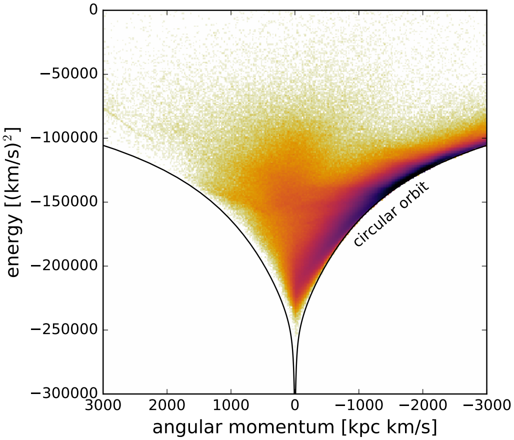
Overview of the loss-cone theory

Eugene Vasiliev



Como workshop on TDEs and EMRIs
5–7 September 2022

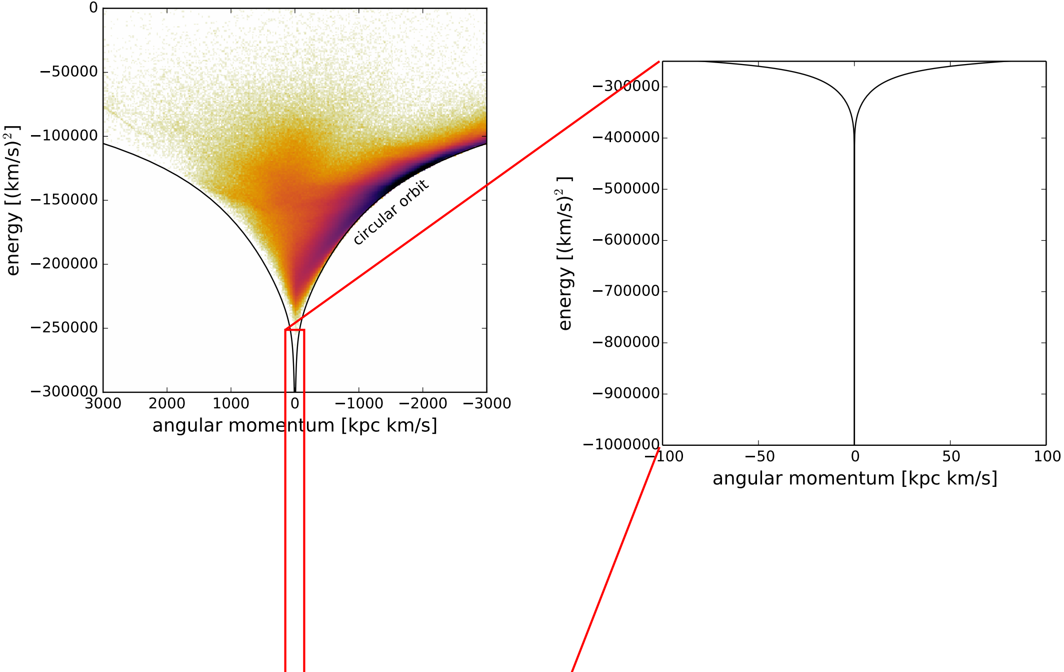
Prelude



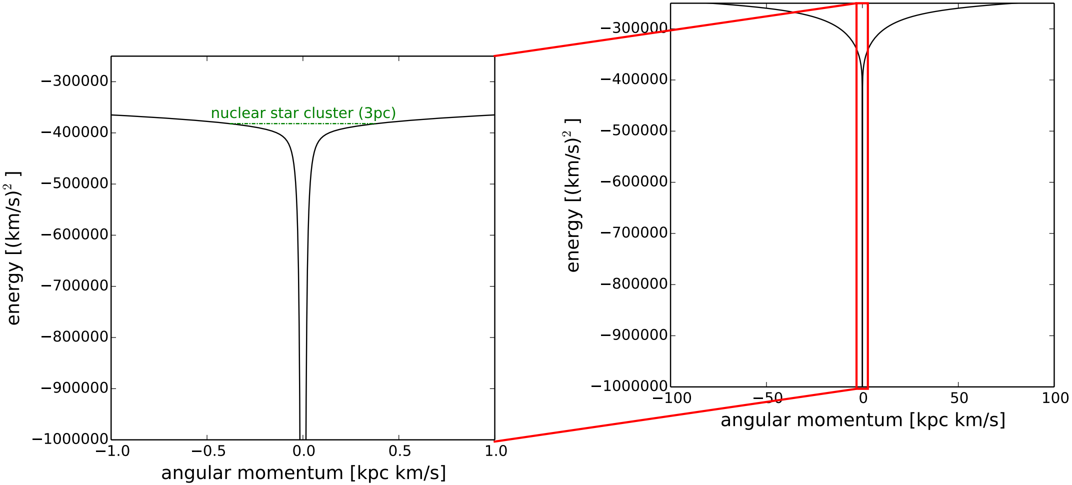
3×10^7 stars in Gaia DR3

with 6d phase-space coordinates

Prelude

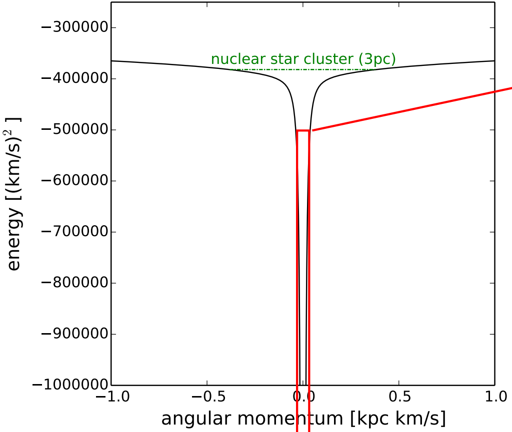

0
−50000
−100000
−150000
−200000
−250000
−300000
energy [(km/s)^2]
3000
2000
1000
0
−1000
−2000
−3000
angular momentum [kpc km/s]
circular orbit
−300000
−400000
−500000
−600000
−700000
−800000
−900000
−1000000
energy [(km/s)^2]
−100
−50
0
50
100
angular momentum [kpc km/s]

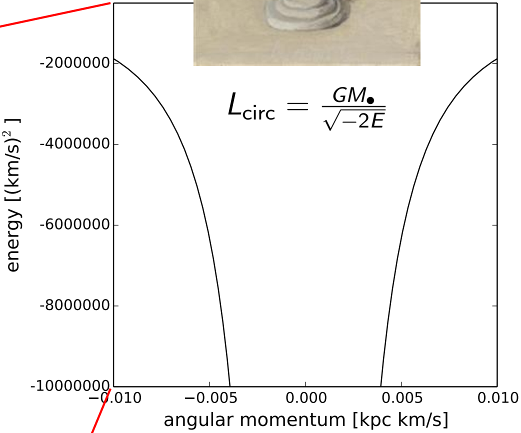
Prelude



nuclear star cluster (3pc)
energy [(km/s)^2]
angular momentum [kpc km/s]
−300000
−400000
−500000
−600000
−700000
−800000
−900000
−1000000
−1.0
−0.5
0.0
0.5
1.0
energy [(km/s)^2]
angular momentum [kpc km/s]
−100
−50
0
50
100

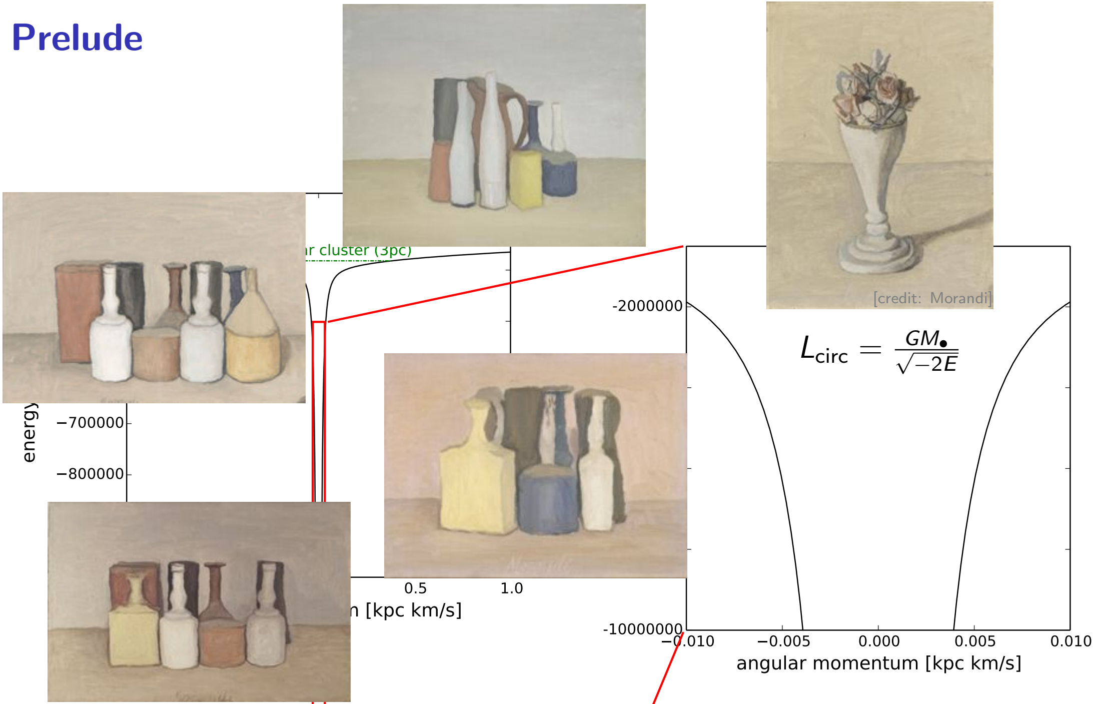
Prelude





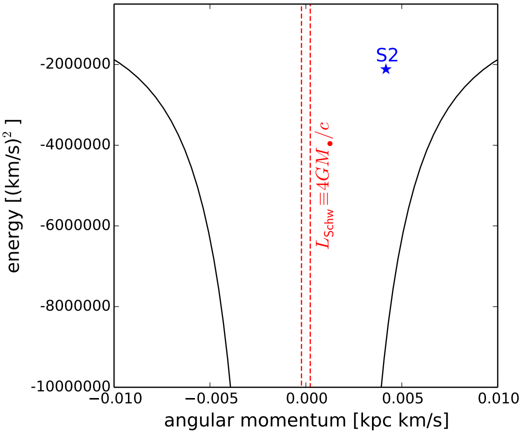
$$L_{\rm circ} = \frac{GM_\bullet}{\sqrt{-2E}}$$

Prelude



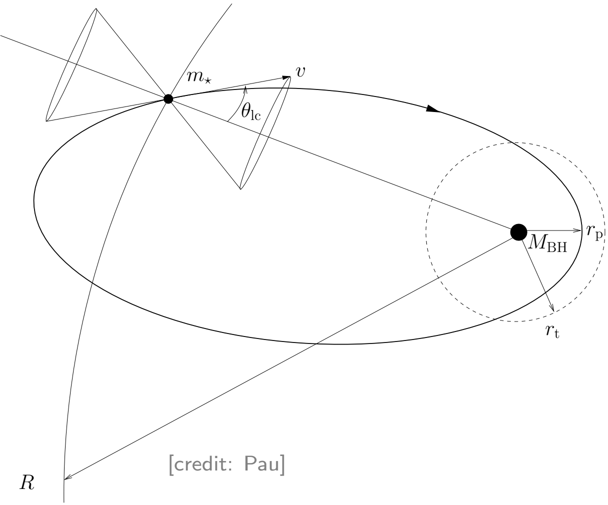
How many times do I need to zoom in to resolve the loss cone?

Prelude

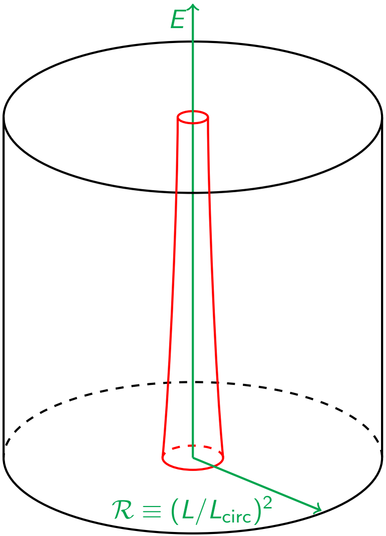


S2
$L_{\rm Schw} \equiv 4GM_\bullet/c$
energy [(km/s)^2]
-2000000
-4000000
-6000000
-8000000
-10000000
−0.010
−0.005
0.000
0.005
0.010
angular momentum [kpc km/s]

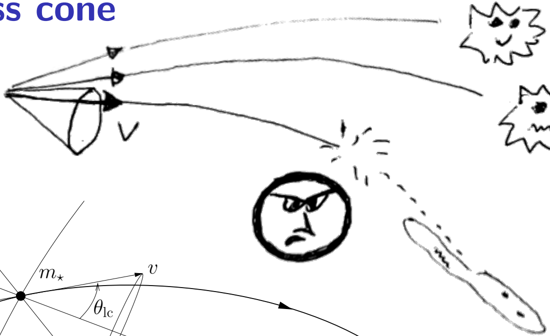
The loss cone

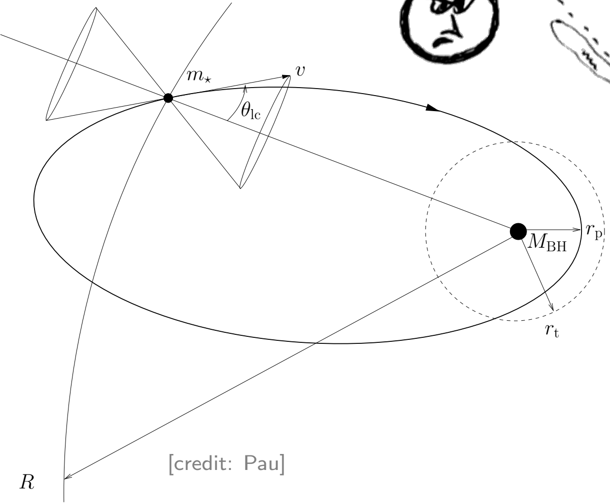


[credit: Pau]

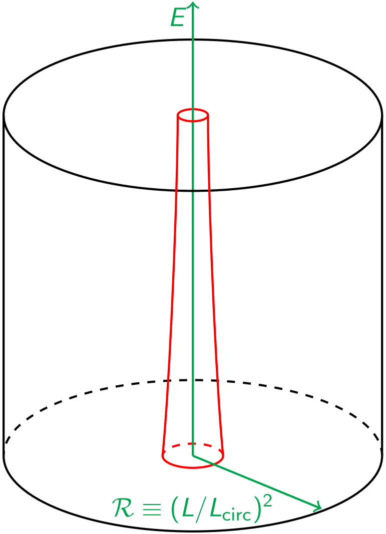


The loss cone





[credit: Pau]



Orbits intro

Keplerian orbit with semimajor axis a and eccentricity e:

$$E = -\frac{GM_\bullet}{2a}$$

$$T_{\rm orb} = \frac{2\pi\, a^{3/2}}{\sqrt{GM_\bullet}}$$

$$L_{\rm circ} = \frac{GM_\bullet}{\sqrt{-2E}} = \sqrt{GM_\bullet a}$$

$$L = L_{\rm circ}\,\sqrt{1-e^2}$$

$$\mathcal{R} \equiv (L/L_{\rm circ})^2 = 1 - e^2 \approx 2(1-e) \;\text{ for very eccentric orbits}$$

If the physical radius of the loss cone $r_{\rm LC} \ll a$, only stars with $1 - e \ll 1$ are able to enter it, and in this case $L_{\rm LC} \approx \sqrt{2\,G\,M_\bullet\,r_{\rm LC}}$.

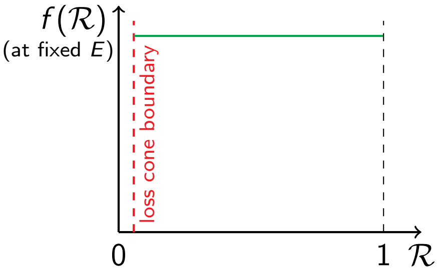
Distribution functions intro

$f(\mathbf{x},\ \mathbf{v})$ is the DF in the 6d phase space (normalized so that $\int f\, d^3\mathbf{x}\, d^3\mathbf{v} = N_\star m_\star$)
according to Jeans' theorem, in a steady state it may depend only on the integrals of motion, i.e., in a spherical potential, $f(E, L)$ or $f(E, \mathcal{R})$.

The mass of stars per unit $E, \mathcal{R}$ is
$N(E,\mathcal{R})\, dE\, d\mathcal{R} = g(E,\mathcal{R})\, f(E,\mathcal{R})\, dE\, d\mathcal{R}$,
where the density of states $g(E,\mathcal{R}) = 4\pi^2\, T_{\rm orb}(E,\mathcal{R})\, L^2_{\rm circ}(E) \approx g(E)$;
in the Keplerian case, $g(E) = \frac{\sqrt{2}\,\pi^3\,(GM_\bullet)^3}{(-E)^{5/2}}$.

In case of isotropic velocity distribution ($\Leftrightarrow$ "thermal" eccentricity distribution), $f(E,\mathcal{R}) = f(E)$.

Such a distribution is thermodynamically preferred, but cannot be fully achieved because of the existence of the loss cone.



Relaxation intro

Classical ("Chandrasekhar") two-body relaxation theory:
under the assumptions of

1. uncorrelated pairwise encounters,
2. weak deflections (impact parameter $b \gg b_{90} \equiv Gm_\star/v^2$),
3. slow evolution ($T_{\rm orb} \ll T_{\rm rel}$),

the evolution of the DF $f(E,\mathcal{R})$ can be described by
the orbit-averaged Fokker–Planck equation:

$$\frac{\partial\big[f(E,\mathcal{R},t)\,g(E,\mathcal{R})\big]}{\partial t} = -\frac{\partial \mathcal{F}_E(E,\mathcal{R},t)}{\partial E} - \frac{\partial \mathcal{F}_\mathcal{R}(E,\mathcal{R},t)}{\partial \mathcal{R}}$$

$$\left.\begin{aligned} -\mathcal{F}_E &= \mathcal{D}_{EE}\frac{\partial f}{\partial E} + \mathcal{D}_{E\mathcal{R}}\frac{\partial f}{\partial \mathcal{R}} + m_\star\,\mathcal{A}_E\,f \\ -\mathcal{F}_\mathcal{R} &= \mathcal{D}_{\mathcal{R}E}\frac{\partial f}{\partial E} + \mathcal{D}_{\mathcal{R}\mathcal{R}}\frac{\partial f}{\partial \mathcal{R}} + m_\star\,\mathcal{A}_\mathcal{R}\,f \end{aligned}\right\}\ \text{fluxes in } E \text{ and } \mathcal{R}$$

diffusion coefficients   stellar mass   advection coefficients

Relaxation intro (2)

Advection and diffusion coefficients are given by some integrals over the DF of *field* stars, and nearly always this field DF is approximated by the isotropized form $\overline{f}(E) \equiv \int_0^1 f(E, \mathcal{R})\, d\mathcal{R}$:

$\mathcal{D}_{EE}(E, \mathcal{R}) = m_\star \int dE'\, \overline{f}(E')\, K(E', E, \mathcal{R})$ with some kernel K.

Usually the field DF is the same as the test stars' DF evolving under the Fokker–Planck eqn.

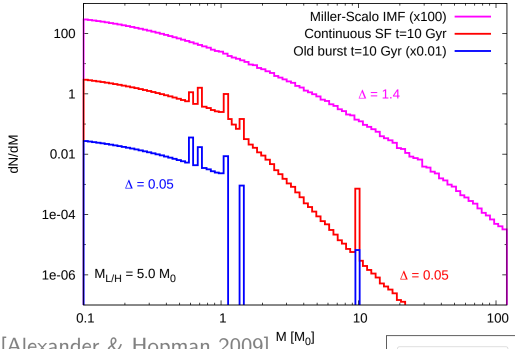
In the multi-mass case (e.g., $1\,M_\odot$ stars and $10\,M_\odot$ black holes), diffusion coefficients are the same for all species, and the field DF is given by the sum of all species' DFs *additionally weighted by field stars' mass*, i.e.,
$\mathcal{D}_{EE}(E, \mathcal{R}) = \sum_i m_{\star,i} \int dE'\, \overline{f}_i(E')\, K(E', E, \mathcal{R})$.
Thus the relaxation rate is often dominated by the most massive species.

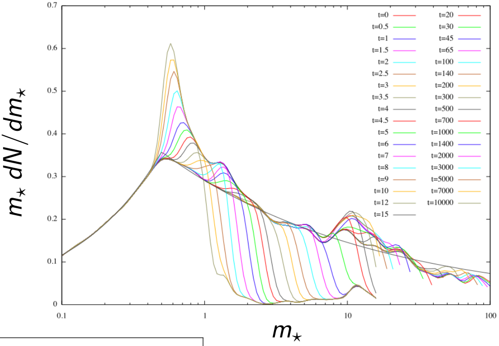
OTOH the advection coefficients are not weighted by the field star masses,
$\mathcal{A}_E = \sum_i \int dE'\, \overline{f}_i(E')\, \text{Ӽ}(E', E, \mathcal{R})$,
but then additionally multiplied by the test star mass in the eqn for flux.

This is what gives rise to dynamical friction and mass segregation.

Relaxation in multimass systems

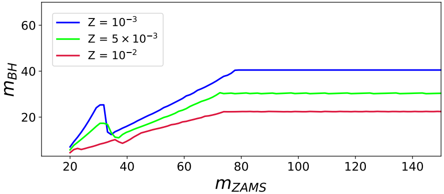
For a typical IMF, a few % of mass is contained in black holes with $m_\star \gtrsim 10\, M_\odot$: this means that they significantly contribute to the relaxation rate even without mass segregation!





[Alexander & Hopman 2009]

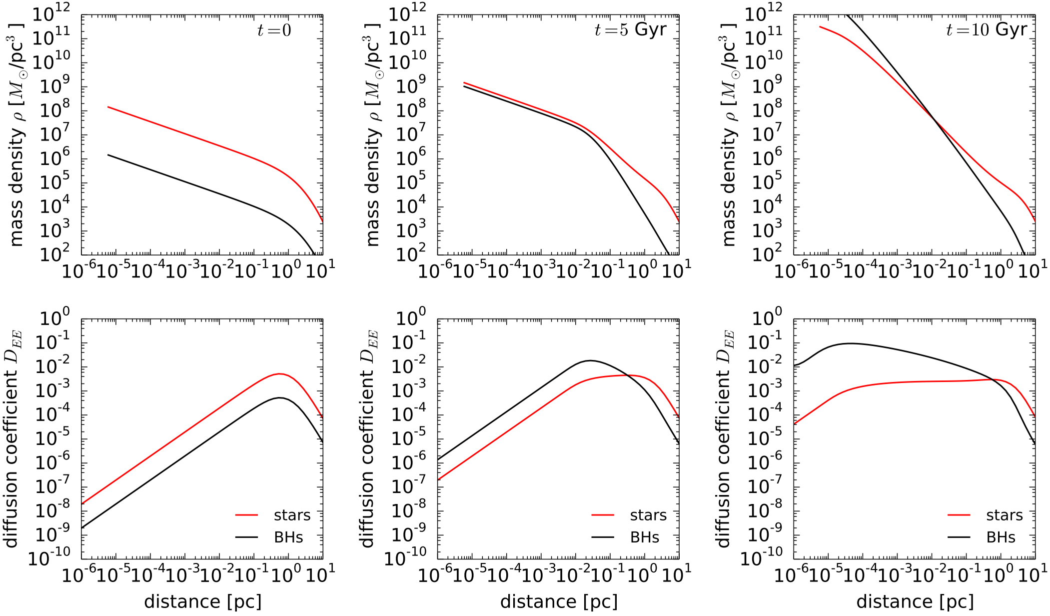
see also
Preto & Amaro-Seoane 2010,
Stone & Metzger 2016, ...



low-metallicity stars produce heavier BHs

Diffusion in energy and mass segregation

example of re-growth of the Bahcall–Wolf cusp



Diffusion in angular momentum

If we ignore for the moment the diffusion in the energy direction, the 1d Fokker–Planck equation for $f(\mathcal{R},t)\big|_{E=\mathrm{const}}$ is

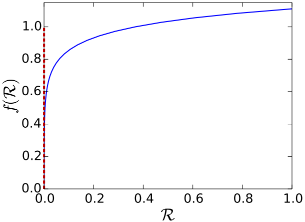
$$\frac{\partial f(\mathcal{R},t)}{\partial t} = -\frac{\partial \mathcal{F}_\mathcal{R}(\mathcal{R},t)}{\partial \mathcal{R}}, \qquad -\mathcal{F}_\mathcal{R} = \mathcal{D}_{\mathcal{R}\mathcal{R}}\frac{\partial f}{\partial \mathcal{R}} + m_\star A_\mathcal{R}\, f$$

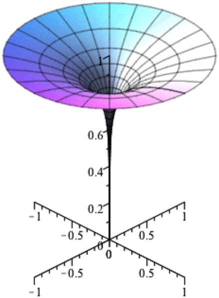
The advection (drift) coefficient $\mathcal{A}_\mathcal{R}$ turns out to be zero (because the flux should vanish for the isotropic DF $f(\mathcal{R}) = \mathrm{const}$), and the diffusion coefficient, to first order, is $\mathcal{D}_{\mathcal{R}\mathcal{R}}(E,\mathcal{R}) \approx \mathscr{D}(E)\,\mathcal{R}$.

This is equivalent to the diffusion or heat conduction equation in the cylindrical geometry, and the steady-state solution is

$0 = \mathscr{D}\,\frac{\partial}{\partial\mathcal{R}}\left(\mathcal{R}\,\frac{\partial f}{\partial \mathcal{R}}\right) \implies$

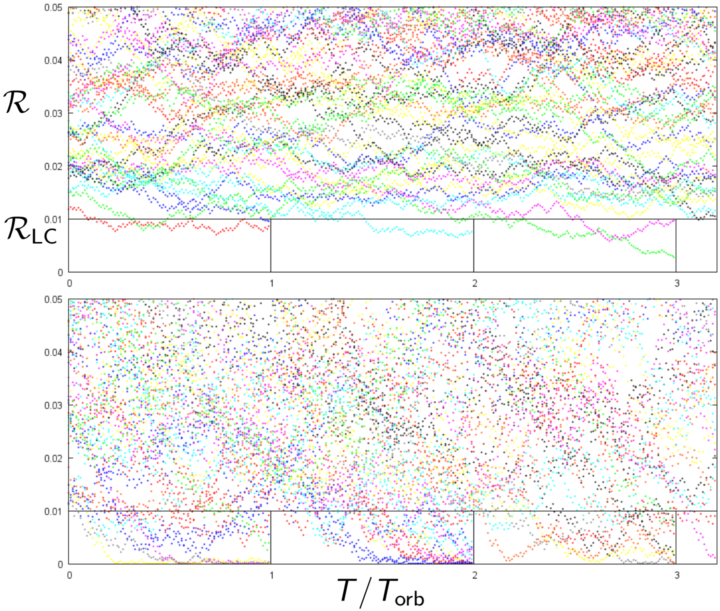
$$f(\mathcal{R}) = \frac{\overline{f}\ \ln[\mathcal{R}/\mathcal{R}_\mathrm{LC}]}{\ln[1/\mathcal{R}_\mathrm{LC}] - 1 + \mathcal{R}_\mathrm{LC}}.$$





Empty vs. full loss cone regimes

recall that stars are captured (loss cone is purged) only at pericentre passages



Two regimes:

compare $T_{\rm orb}$ with the loss cone repopulation timescale $\sqrt{\mathcal{D}_{\mathcal{RR}}\,T_{\rm rep}} \simeq R_{\rm LC}$.

$$q \equiv \frac{T_{\rm orb}}{T_{\rm rep}} = \frac{\mathscr{D}\,T_{\rm orb}}{R_{\rm LC}}.$$

$q \ll 1$: empty loss cone

$q \gg 1$: full loss cone

Empty vs. full loss cone regimes

When the loss cone is not entirely empty, the DF at its boundary is > 0; one can write the boundary condition in the form

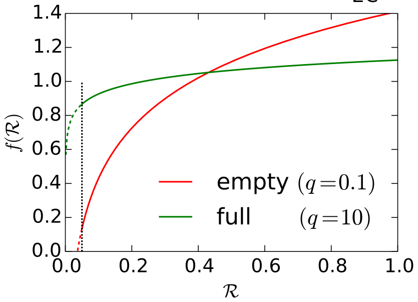
$$0 = f(\mathcal{R}_{\rm LC}) - \alpha\,\mathcal{R}\,\frac{\partial f(\mathcal{R})}{\partial \mathcal{R}}\bigg|_{\mathcal{R}_{\rm LC}}, \quad \text{where}\ \ \alpha \approx (q^2+q^4)^{1/4} \text{ and } q = \frac{\mathscr{D}\,T_{\rm orb}}{R_{\rm LC}}.$$

The steady-state solution is

$$f(\mathcal{R}) = \frac{\overline{f}\ \big(\ln[\mathcal{R}/\mathcal{R}_{\rm LC}] + \alpha\big)}{\ln[1/\mathcal{R}_{\rm LC}] + (1-\mathcal{R}_{\rm LC})(\alpha-1)},$$

and the flux into the loss cone is

$$-\mathcal{F}_{\mathcal{R}} = \frac{\overline{f}\ \mathscr{D}}{\ln[1/\mathcal{R}_{\rm LC}] + (1-\mathcal{R}_{\rm LC})(\alpha-1)}.$$



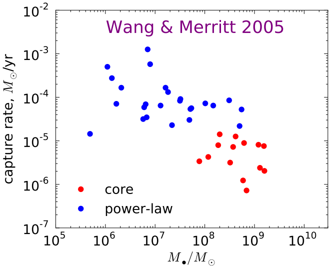
In the empty LC regime, the flux is proportional to the relaxation rate $\mathscr{D}$ and only logarithmically depends on the loss-cone size $\mathcal{R}_{\rm LC}$, while in the full LC regime the flux is linearly proportional to $\mathcal{R}_{\rm LC}$ and nearly independent of the relaxation rate.

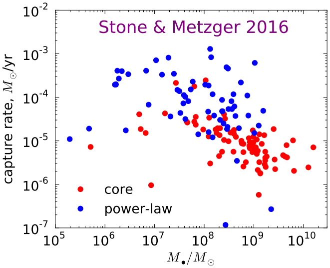
Dependence of capture rates on galaxy properties

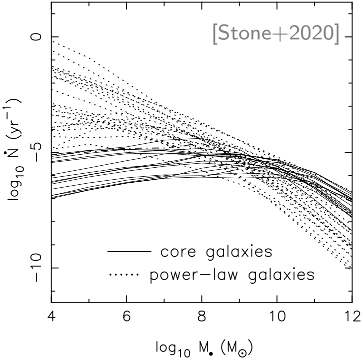
Local relaxation time $T_{\rm rel}(r) \equiv \dfrac{0.34\,\sigma(r)^3}{G^2\,\rho(r)\,m_\star\,\ln\Lambda} \sim \mathscr{D}^{-1}$.

In the Keplerian potential $(r \lesssim r_{\rm infl})$, $\sigma(r) \propto \sqrt{GM_\bullet/r}$, and if we assume $\rho(r) \propto \dfrac{M_\bullet}{r_{\rm infl}^3}\left(\dfrac{r}{r_{\rm infl}}\right)^{-\gamma}$, then

$$T_{\rm rel} \propto \frac{M_\bullet}{m_\star}\,\frac{r_{\rm infl}^{3/2}}{\sqrt{GM_\bullet}}\left(\frac{r}{r_{\rm infl}}\right)^{\gamma-3/2} \simeq \frac{M_\bullet}{m_\star}\left(\frac{r}{r_{\rm infl}}\right)^{\gamma-3/2} T_{\rm orb}(r_{\rm infl}).$$

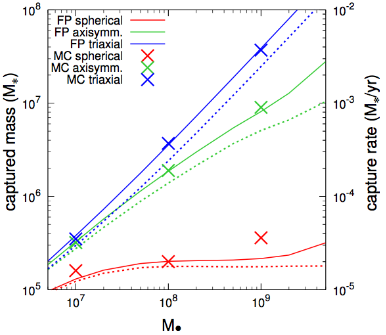




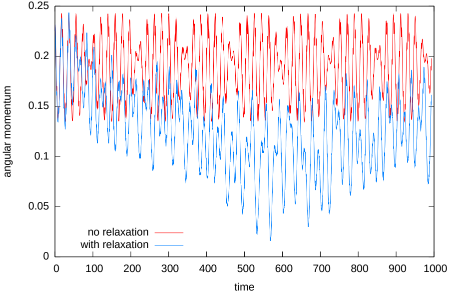


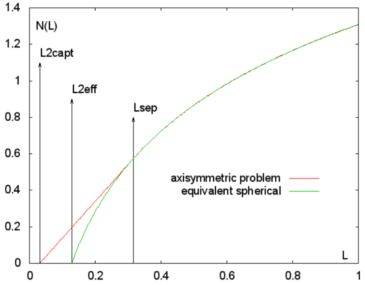
Complication #1: non-spherical galaxy potentials

Even if the black hole dominates the total potential, a non-spherical stellar distribution produces torques that lead to periodic variations of orbital angular momentum even in absense of relaxation.



Stars from the "centrophilic" orbits can sustain much higher capture rates than in spherical galaxies *if the relaxation rate is low*, and when they are drained, the capture rates are only moderately higher due to log-dependence of flux on $\mathcal{R}_{\rm LC}$.

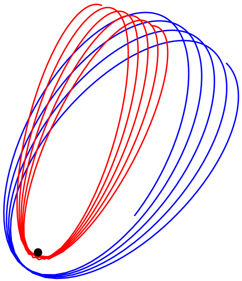


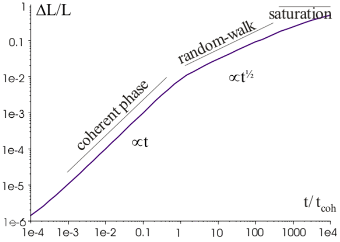


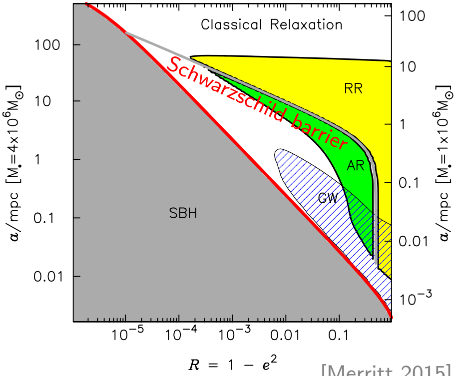
Complication #2: resonant relaxation

In a [nearly-]Keplerian potential, orbits are almost closed ellipses and can interact with each other over many periods before "decorrelating" due to orbit precession.

This gives rise to enhanced relaxation in angular momentum [Rauch & Tremaine 1996; Hopman & Alexander 2006], but only at eccentricities below the "Schwarzschild barrier" set by relativistic precession [Merritt+ 2011; Brem+ 2013; Hamers+ 2014; Bar-Or & Alexander 2015].



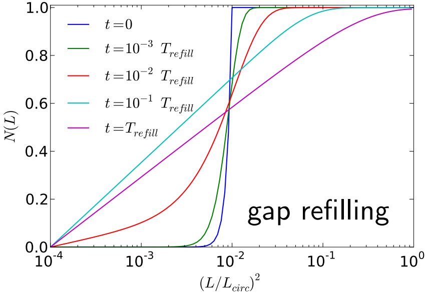


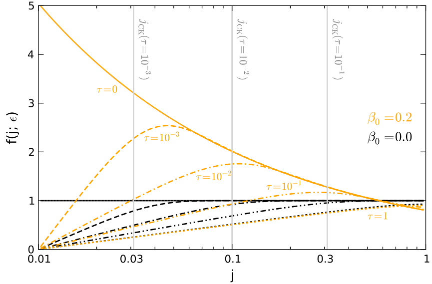


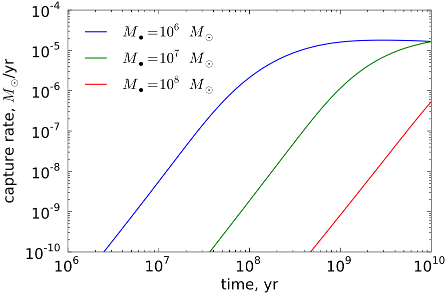
[Merritt 2015]

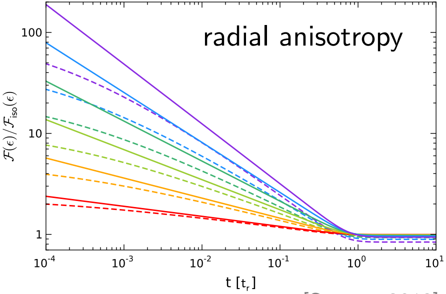
Complication #3: anisotropic and time-dependent loss cones

steady-state log profile $f(\mathcal{R})$ is established only after [a fraction of] $T_{\rm rel}$







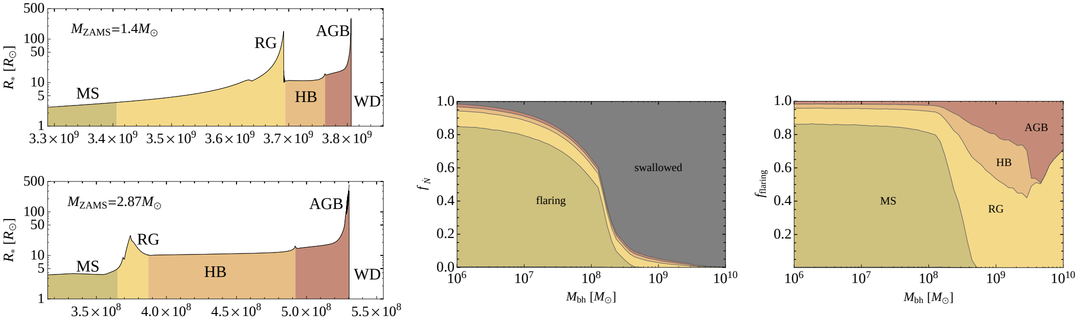


[Merritt & Wang 2005; Lezhnin & Vasiliev 2015]

[Stone+ 2018]

Complication #4: giant stars and partial disruptions

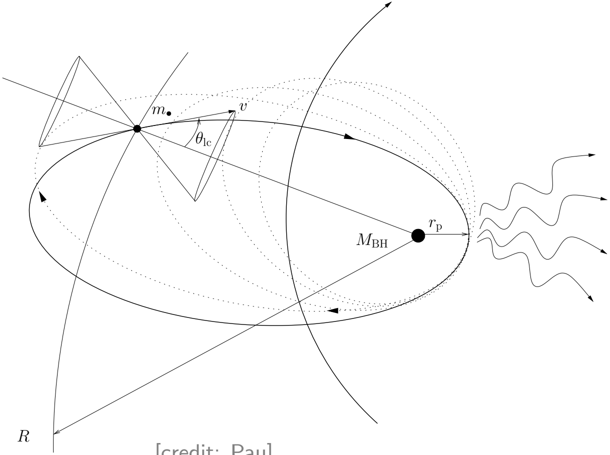
Since the tidal radius is $r_{\rm LC} \simeq r_\star (M_\bullet/m_\star)^{1/3}$, giant stars can have a significant contribution to the total TDE rate [Magorrian & Tremaine 1999, McLeod+ 2012]. Stars can "grow" into the loss cone even without changing their orbit [Syer & Ulmer 1999], and the outer envelope of a giant can be repeatedly stripped in many partial disruption flares [McLeod+ 2013].
For $M_\bullet \gtrsim 10^8\,M_\odot$, $r_{\rm LC}$ for main-sequence stars is below $r_{\rm Schw}$, so only giants produce TDE flares.



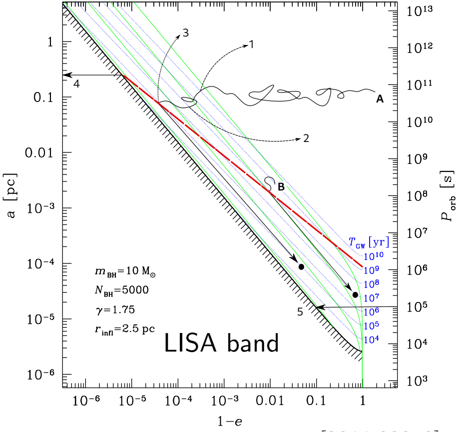
[McLeod+ 2012]

Tidal disruptions vs. EMRIs

Compact objects (NS, BH) are not tidally disrupted, but can lose enough energy to gravitational waves during close pericentre passages to end up on very tight orbits (hence in the LISA frequency band), possibly completing hundreds of orbits before merging.



[credit: Pau]



[2011.03059]

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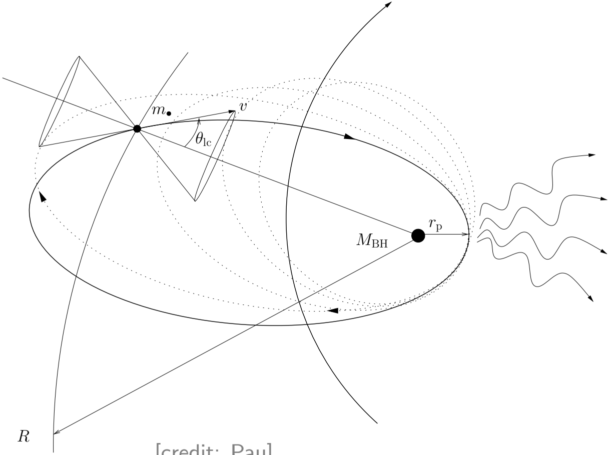
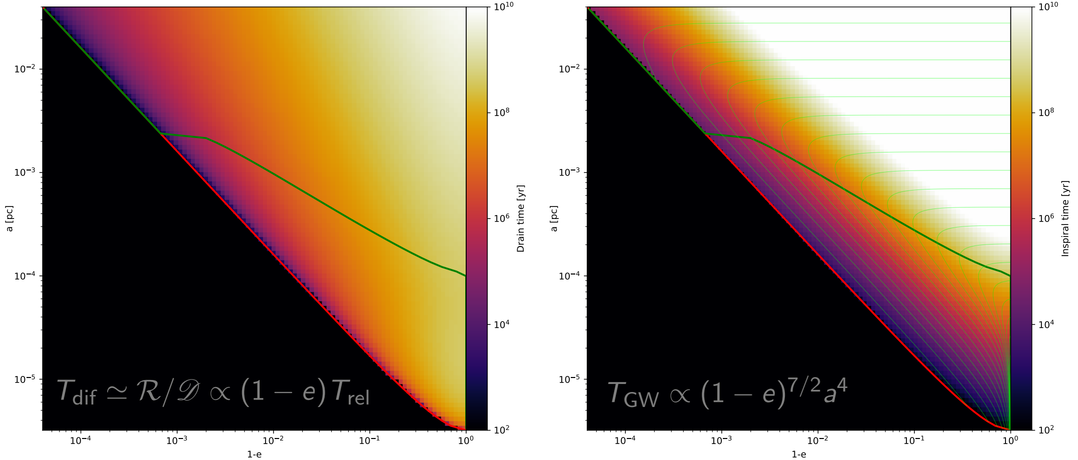


[credit: Pau]

[credit: Giacomo Balla]

Formation of EMRIs

In the standard scenario, COs diffuse through phase space just like stars, but when they reach high enough eccentricity that the GW emission timescale $T_{\rm GW}$ becomes shorter than the diffusion timescale, they slide down towards small a and low e.
The division line $T_{\rm GW} = T_{\rm dif}$ crosses the loss-cone boundary at some $a_{\rm GW}$.



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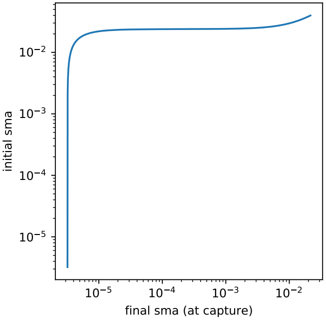
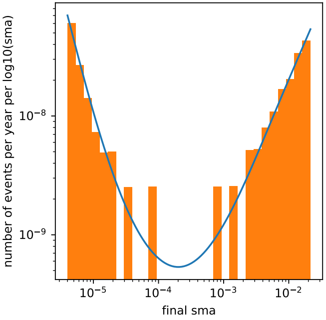
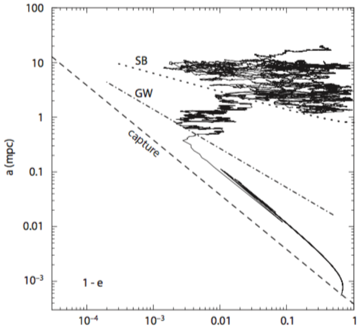
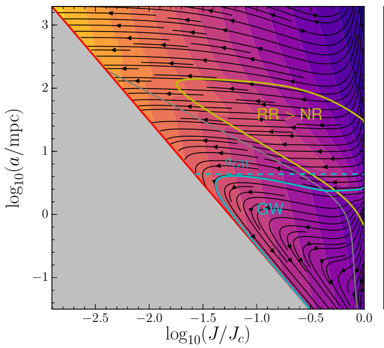


Resonant relaxation was once thought to be a significant factor affecting the EMRI rates, but more careful analysis showed that it is likely unimportant in the region of interest [cf. Alexander 2017]: the GW onset occurs to the left of the Schwarzschild barrier, and hence still determined by ordinary two-body relaxation.





[Merritt+ 2010]



[Bar-Or & Alexander 2015]

Summary

early loss-cone theory

Summary

early loss-cone theory

modern loss-cone theory