Modern stellar dynamics, lecture 2: density and projection

Eugene Vasiliev

Institute of Astronomy, Cambridge

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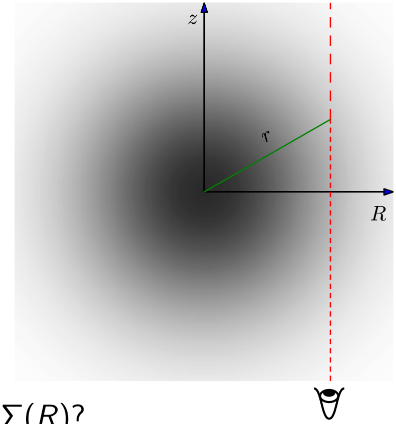
Surface density

For spherical systems: 3d density is $\rho(r)$, projected density is

$$\Sigma(R) = \int_{-\infty}^{\infty} \mathrm{d}z\ \rho(\sqrt{R^2 + z^2})$$
$$= 2\int_{R}^{\infty} \mathrm{d}r\ \frac{r}{\sqrt{r^2 - R^2}}\ \rho(r).$$



How to invert this equation to obtain $\rho(r)$ from $\Sigma(R)$?

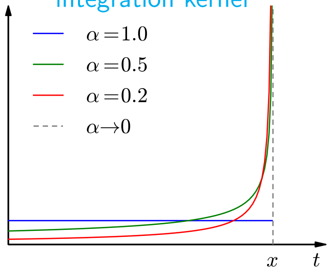
Fractional calculus

Ordinary integral: $\mathcal{J}\left[f(x)\right] \equiv \int_0^x \mathrm{d}t\ f(t)$.

Generalization to an arbitrary real number $\alpha > 0$ (Riemann–Liouville integral):

$$\mathcal{J}^{\alpha}\left[f(x)\right] \equiv \frac{1}{\Gamma(\alpha)} \int_0^x \mathrm{d}t\ f(t)\,(x-t)^{\alpha-1}.$$

integration kernel



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$$\mathcal{J}^\alpha\left[f(x)\right] \equiv \frac{1}{\Gamma(\alpha)} \int_0^x \mathrm{d}t\ f(t)\,(x-t)^{\alpha-1}.$$

Works as expected for monomials (even for non-integer n)

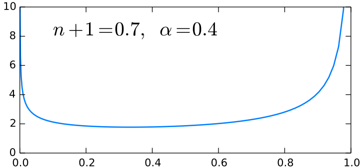
$$f(x) = \frac{x^n}{n!} \equiv \frac{x^n}{\Gamma(n+1)}, \qquad \mathcal{J}\left[f(x)\right] = \frac{x^{n+1}}{(n+1)!} \equiv \frac{x^{n+1}}{\Gamma(n+2)}.$$

Fractional integral: $\quad \mathcal{J}^\alpha\left[f(x)\right] = \frac{1}{\Gamma(n+1)\,\Gamma(\alpha)} \int_0^x \mathrm{d}t\ t^n\,(x-t)^{\alpha-1}$,

substituting $t = u\,x$, $\quad \mathcal{J}^\alpha\left[f(x)\right] = \frac{x^{n+\alpha}}{\Gamma(n+1)\,\Gamma(\alpha)} \int_0^1 \mathrm{d}u\ u^n\,(1-u)^{\alpha-1} =$

$$\frac{x^{n+\alpha}}{\Gamma(n+1)\,\Gamma(\alpha)} \frac{\Gamma(n+1)\,\Gamma(\alpha)}{\Gamma(n+\alpha+1)}.$$

Beta-function $\mathrm{B}(n+1,\alpha)$



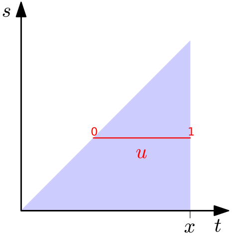
Fractional calculus

Fractional integration is commutative and distributive:

$$\mathcal{J}^{\alpha}\Big[\mathcal{J}^{\beta}\big[f(x)\big]\Big] = \mathcal{J}^{\beta}\Big[\mathcal{J}^{\alpha}\big[f(x)\big]\Big] = \mathcal{J}^{\alpha+\beta}\big[f(x)\big].$$

Proof:

$$\mathcal{J}^{\alpha}\Big[\mathcal{J}^{\beta}\big[f(x)\big]\Big] = \frac{1}{\Gamma(\alpha)\,\Gamma(\beta)}\int_0^x \mathrm{d}t\,(x-t)^{\alpha-1}\int_0^t \mathrm{d}s\,(t-s)^{\beta-1}\,f(s) =$$



exchanging the order of integration

$$\frac{1}{\Gamma(\alpha)\,\Gamma(\beta)}\int_0^s \mathrm{d}s\,f(s)\int_s^x \mathrm{d}t\,(x-t)^{\alpha-1}\,(t-s)^{\beta-1} =$$

substituting $t = s + u\,(x-s)$

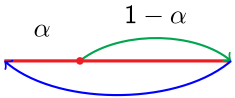
$$\frac{1}{\Gamma(\alpha)\,\Gamma(\beta)}\int_0^s \mathrm{d}s\,f(s)\,(x-s)^{\alpha+\beta-1}\int_0^1 \mathrm{d}u\,(1-u)^{\alpha-1}\,u^{\beta-1} =$$

$$\frac{1}{\Gamma(\alpha)\,\Gamma(\beta)}\int_0^s \mathrm{d}s\,f(s)\,(x-s)^{\alpha+\beta-1}\underbrace{\frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}}_{=\,\mathrm{B}(\alpha,\beta)} = \mathcal{J}^{\alpha+\beta}\big[f(x)\big].$$

Fractional calculus

Fractional derivative is defined through fractional integral:

$$\mathcal{D}^{\alpha}\left[f(x)\right] \equiv \frac{\mathrm{d}}{\mathrm{d}x}\,\mathcal{J}^{1-\alpha}\left[f(x)\right] \;=\; \frac{1}{\Gamma(1-\alpha)}\,\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \mathrm{d}t\,\frac{f(t)}{(x-t)^{\alpha}}.$$



$(0 < \alpha < 1)$

As expected, $\mathcal{D}^{\alpha}\Big[\mathcal{J}^{\alpha}\left[f\right]\Big] = f$, in other words,

if $\quad g(x) = \mathcal{J}^{\alpha}\left[f\right] = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^x \mathrm{d}t\,\frac{f(t)}{(x-t)^{1-\alpha}},$

then $\quad f(t) = \mathcal{D}^{\alpha}\left[g\right] = \dfrac{1}{\Gamma(1-\alpha)}\,\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_0^t \mathrm{d}x\,\frac{g(x)}{(t-x)^{\alpha}}.$

Likewise, if $\quad g(x) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_x^{\infty} \mathrm{d}t\,\frac{f(t)}{(t-x)^{1-\alpha}},$

then $\quad f(t) = -\dfrac{1}{\Gamma(1-\alpha)}\,\dfrac{\mathrm{d}}{\mathrm{d}t}\displaystyle\int_t^{\infty} \mathrm{d}x\,\frac{g(x)}{(x-t)^{\alpha}}.$

} Abel integral equation

Deprojection of spherical density profiles

$$\Sigma(R) = 2\int_0^\infty \mathrm{d}z\ \rho(\sqrt{R^2 + z^2})$$

substituting $s = R^2,\ t = r^2 = R^2 + z^2$, we have

$$\Sigma(s) = \int_s^\infty \mathrm{d}t\ \frac{\rho(t)}{(t - s)^{1/2}},$$

this is the Abel integral equation with $\alpha = 1/2$, hence

$$\rho(t) = -\frac{1}{\Gamma(\alpha)\,\Gamma(1-\alpha)}\ \frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty \mathrm{d}s\ \frac{\Sigma(s)}{(s-t)^\alpha}$$

$$= -\frac{1}{\Gamma(\alpha)\,\Gamma(1-\alpha)}\int_t^\infty \mathrm{d}s\ \frac{1}{(s-t)^\alpha}\ \frac{\mathrm{d}\Sigma(s)}{\mathrm{d}s}.$$

$$= \frac{\sin \pi\alpha}{\pi}$$

$$\rho(r) = -\int_r^\infty \mathrm{d}R\ \frac{1}{\pi\sqrt{R^2 - r^2}}\frac{\mathrm{d}\Sigma(R)}{\mathrm{d}R}.$$

Examples of spherical density profiles

1. Power law: $\Sigma(R) \propto R^{-n} \implies \rho(r) \propto \int_r^\infty \mathrm{d}R \, \frac{n\,R^{-n-1}}{\pi\sqrt{R^2-r^2}} \propto r^{-n-1}$.
2. Plummer model: $\Sigma(R) = \frac{1}{\pi}\frac{M\,a^2}{(r^2+a^2)^2}, \quad \rho(r) = \frac{3}{4\pi}\frac{M\,a^2}{(r^2+a^2)^{5/2}}$.
3. Dehnen model: $\rho(r) = \frac{3-\gamma}{4\pi}\frac{M\,a}{r^\gamma\,(r+a)^{4-\gamma}}$ ($\gamma=1$: Hernquist, $\gamma=2$: Jaffe).
4. Navarro–Frenk–White: $\rho(r) = \frac{1}{4\pi}\frac{M}{r\,(r+a)^2}$.
5. General double-power-law model [Zhao 1995]:

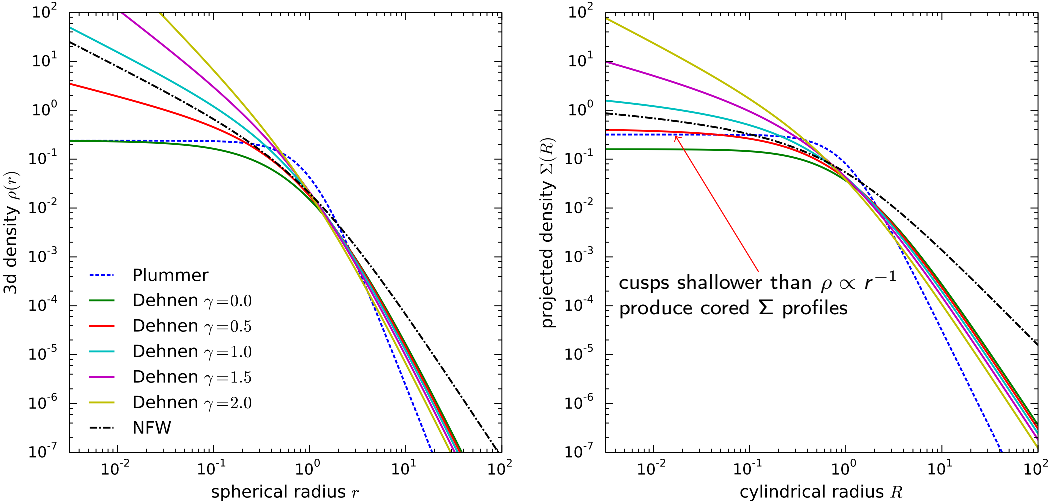
   $\rho(r) = \rho_0\,\left(r/a\right)^{-\gamma}\left(1+[r/a]^\alpha\right)^{(\gamma-\beta)/\alpha}$

   γ – inner slope, β – outer slope, α – transition steepness.

   (when $\beta \le 3$, total mass is infinite)

γ	β	α	
0	5	2	Plummer
γ	4	1	Dehnen
1	3	1	NFW

Examples of spherical density profiles



Examples of spherical density profiles

6. Gaussian profile: $\rho(r) = \frac{M}{(\sqrt{2\pi}\,a)^3} \exp\left[-\frac{1}{2}(r/a)^2\right],$

$$\Sigma(R) = \frac{M}{(\sqrt{2\pi}\,a)^2} \exp\left[-\frac{1}{2}(R/a)^2\right].$$

7. Sérsic profile: $\Sigma(R) = \Sigma_0 \exp\left[-b_n\,(R/a)^{1/n}\right], \quad b_n \approx 2n - 1/3.$

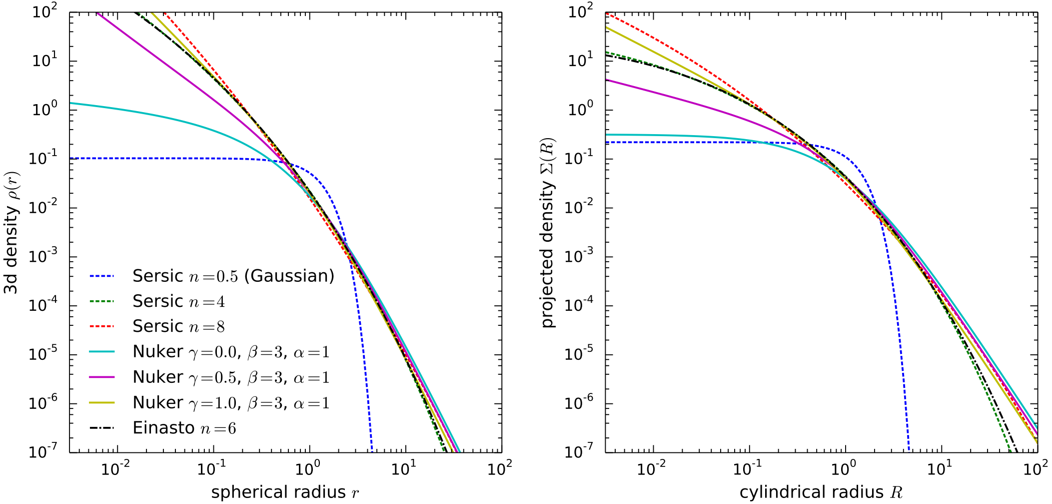
8. Einasto profile: $\rho(r) = \rho_0 \exp\left[-2n\,(r/a)^{1/n}\right].$

Both Sérsic and Einasto are equivalent to a Gaussian profile for $n = 0.5$, and are steeper in the centre and shallower in the outer part for higher values of n.

9. Nuker profile: $\Sigma(R) = \Sigma_0 \left(R/a\right)^{-\gamma} \left(1 + [R/a]^\alpha\right)^{(\gamma-\beta)/\alpha}.$

With a few exceptions, either ρ or Σ is not expressible in elementary functions.
Observers prefer profiles with simple Σ, theorists – with simple ρ.

Examples of spherical density profiles



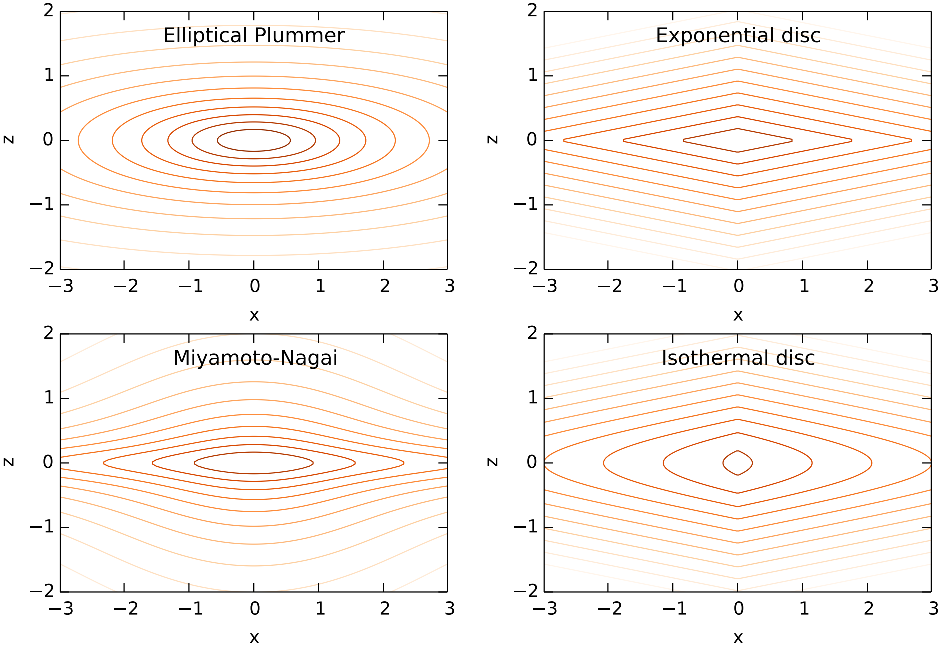
Examples of axisymmetric density profiles

1. Any spherical profile $\rho_{1d}(r)$ squashed along the z axis: $\rho_{2d}(R,z) = \rho_{1d}(m)$, where $m \equiv \sqrt{R^2 + (z/q)^2}$ is the elliptical radius and q is the axis ratio.

2. Miyamoto–Nagai disc: $\rho(R,z) = \frac{1}{4\pi G}\nabla^2\Phi(R,z)$, with
$$\Phi(R,z) = -\frac{G\,M}{\sqrt{R^2 + \left(\sqrt{z^2+b^2}+a\right)^2}}$$
(for $a=0$ this corresponds to a spherical Plummer model).

3. Separable 2d profiles $\rho(R,z) = \Sigma(R)\,h(z)$ with $\int_{-\infty}^{\infty} h(z)\,\mathrm{d}z = 1$,
e.g., an exponential disc: $\Sigma(R) = \dfrac{M}{2\pi\,a^2}\exp\left(-R/a\right)$ with a vertical profile $h(z) = \frac{1}{2b}\exp\left(-|z|/b\right)$ (exponential) or $h(z) = \frac{1}{2b}\mathrm{sech}^2\left(-|z|/b\right)$ (isothermal).

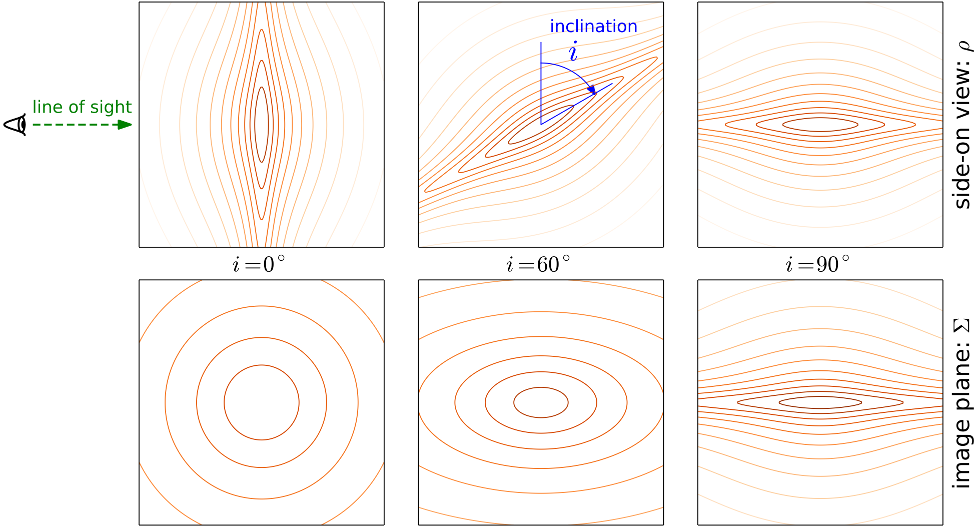
Examples of axisymmetric density profiles



Deprojection of non-spherical systems

$\Sigma(X,Y) \implies \rho(x,y,z)$ – obviously is not unique in a general case.

How about axisymmetric systems? $\rho(R,z) \Leftrightarrow \Sigma(X,Y)$



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How about axisymmetric systems? $\rho(R,z) \Leftrightarrow \Sigma(X,Y)$

Deprojection is unique in the edge-on case ($i=90^\circ$):

let x, z be the image-plane axes, and integrate along the line of sight y

$$\Sigma(x,z) = \int_{-\infty}^{\infty} \mathrm{d}y\; \rho\big(\sqrt{x^2+y^2}, z\big) :$$

for a fixed z, same Abel integral equation as in the spherical case, and

$$\rho(R,z) = -\int_R^{\infty} \mathrm{d}x\, \frac{1}{\pi\sqrt{x^2-R^2}} \frac{\partial \Sigma(x,z)}{\partial x}.$$

On the other hand, in the face-on case ($i=0^\circ$), the projected density $\Sigma(x,y)$ is always round, and we have no information about the distribution of $\rho(R,z)$ along the line of sight z.

Thus we can expect that deprojection at intermediate angles can be non-unique.

Deprojection of axisymmetric density profiles

It turns out that there is a large family of "konus density" profiles that are completely invisible at any inclination $i \leq i_{\mathrm{min}} < 90^\circ$ [Gerhard & Binney 1996; Kochanek & Rybizki 1996].

The Fourier transform of a konus density is identically zero outside a "cone of ignorance" in the Fourier space with opening angle $90^\circ - i_{\mathrm{min}}$ around the k_z axis, and can have arbitrary values inside this cone.

To understand this phenomenon, consider the projected density in the image plane Σ and its Fourier transform $\mathscr{F}_\Sigma$, which are related to the 3d density ρ and its Fourier transform $\mathscr{F}_\rho$ by the projection–slice theorem.

Projection–slice theorem

N-dimensional function $\rho(\mathbf{x}_N)$

$N-1$-dimensional projection $\Sigma(\mathbf{x}_{N-1})$ integrated along direction z

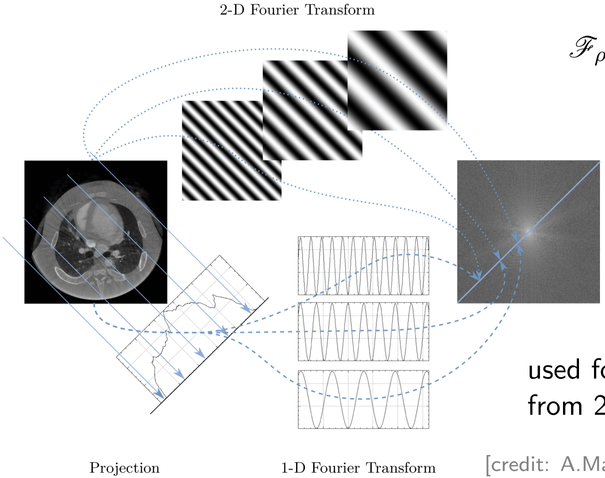
Fourier transform of Σ: $\mathscr{F}_\Sigma(\mathbf{k}_{N-1})$

identical

Fourier transform of ρ: $\mathscr{F}_\rho(\mathbf{k}_N)$

slice through the Fourier space at $k_z = 0$ (multiply by $\delta(k_z)$)

$$\mathscr{F}_\rho(\mathbf{k}) \equiv \int \mathrm{d}^N k \; \rho(\mathbf{x}) \; \exp(\mathrm{i}\,\mathbf{k} \cdot \mathbf{x})$$



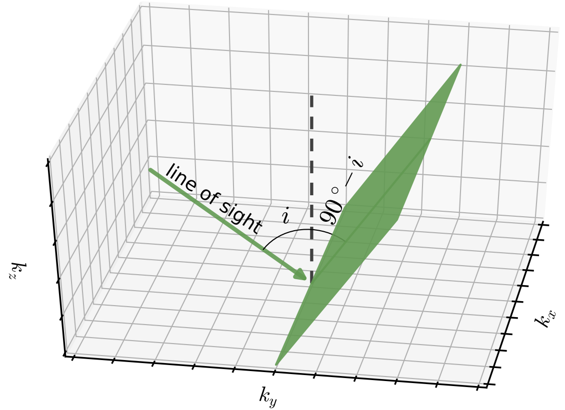
used for reconstruction of 3d shapes from 2d projections (tomography)

[credit: A.Mayer]

Konus density

Consider an axisymmetric density profile ρ projected at an inclination angle i, whose surface density Σ is identically zero.

The Fourier transform $\mathscr{F}_\Sigma$ is zero in a 2d slice of the 3d Fourier space of the original density $\mathscr{F}_\rho$, i.e., in a plane orthogonal to the direction of projection $\mathbf{k}_{\mathrm{LoS}}$.



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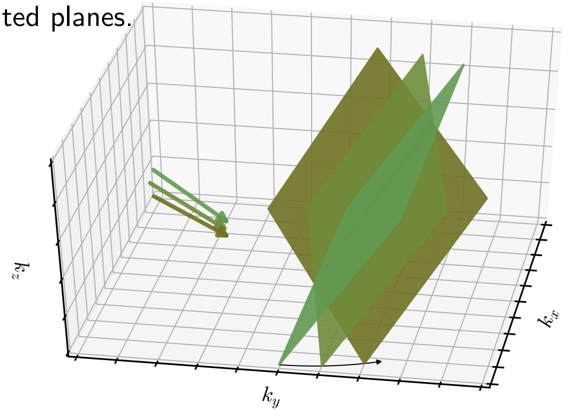

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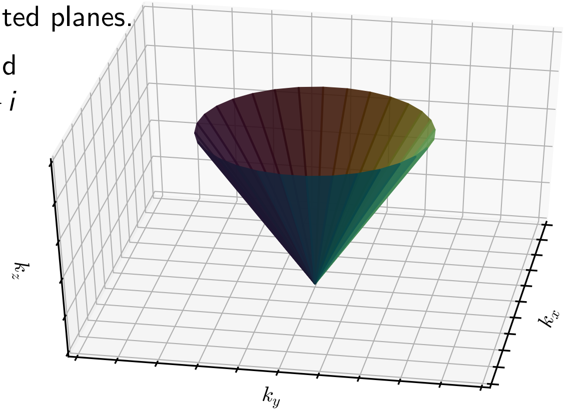
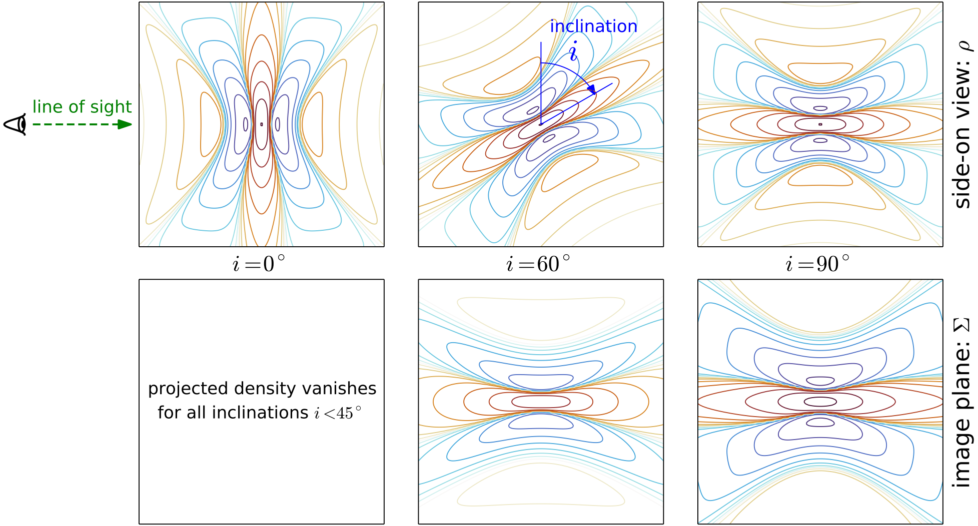
But the values of $\mathscr{F}_\rho$ in a cone around the $\mathbf{k}_z$ axis with opening angle $90^\circ - i$ are unconstrained.



Konus density in practice

Example of a konus density that is invisible at all inclinations $i < 45°$

[from Gerhard&Binney 1996].

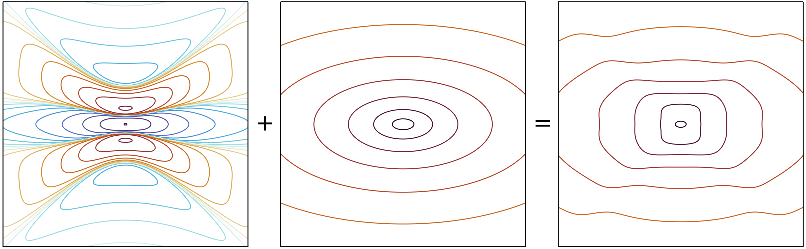


Konus density in practice

Example of a konus density that is invisible at all inclinations $i < 45^\circ$

[from Gerhard&Binney 1996].

Adding it to an ordinary ellipsoidal density profile, one can make it boxy or disky, while still appearing perfectly elliptical in projection.



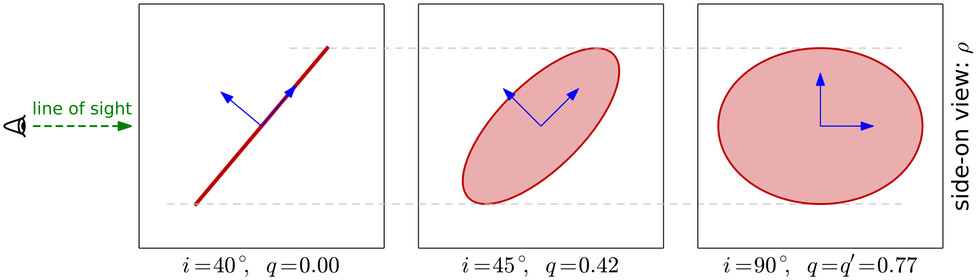
Projection of ellipsoidal profiles

Theorem [Contopoulos 1956; Stark 1977]:
any density profile of the form

$$\rho(x, y, z) = \rho(m), \quad m \equiv \sqrt{x^2 + (y/p)^2 + (z/q)^2},$$

whose isodensity surfaces are ellipsoids with constant axis ratios p, q,
is also stratified on concentric ellipsoids in any projection.

In the axisymmetric case, the projected axis ratio $q' = \sqrt{q^2 \sin^2 i + \cos^2 i}$
$\Longrightarrow$ deprojection is possible for inclination angles $i > i_{\min} \equiv \arccos q'$.



Rotation matrix and Euler angles

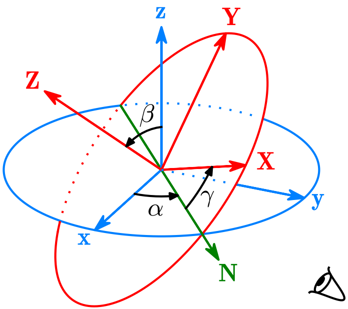
projected intrinsic coordinates

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \mathrm{R} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\mathrm{R} \equiv \begin{pmatrix} c_\alpha c_\gamma - s_\alpha c_\beta s_\gamma & s_\alpha c_\gamma + c_\alpha c_\beta s_\gamma & s_\beta s_\gamma \\ -c_\alpha s_\gamma - s_\alpha c_\beta c_\gamma & -s_\alpha s_\gamma + c_\alpha c_\beta c_\gamma & s_\beta c_\gamma \\ s_\alpha s_\beta & -c_\alpha s_\beta & c_\beta \end{pmatrix}$$

$$c_\circ \equiv \cos\circ, \quad s_\circ \equiv \sin\circ$$

R is an orthogonal matrix, i.e., $\mathrm{R}^{-1} = \mathrm{R}^T$



α, β, γ are Euler angles

(β is the inclination angle)

Projection matrix P is simply the first two rows of R, so a quadratic form defining the 3d equidensity surfaces of ellipsoidally stratified ρ corresponds to another quadratic form for the ellipsoidal isolines of constant Σ (by a series of matrix multiplications).

Projection of triaxial ellipsoidal profiles

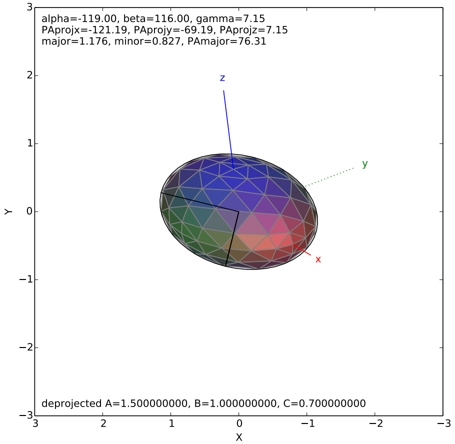
Euler angles α, β, γ

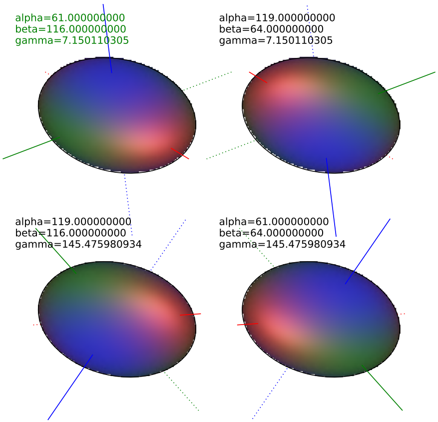
intrinsic axes a, b, c

projected axes a', b'
and position angle η

Given any two triplets, one can determine the third one (projected + intrinsic $\Rightarrow$ 4 possible orientations; projected + angles $\Rightarrow$ intrinsic only in a limited range of angles).

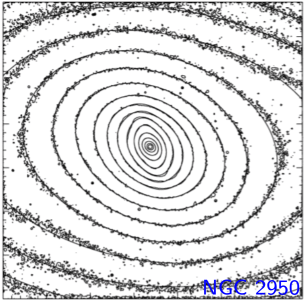
Principal axes of the projected ellipsoid can be misaligned with the projection of any of the three intrinsic principal axes.

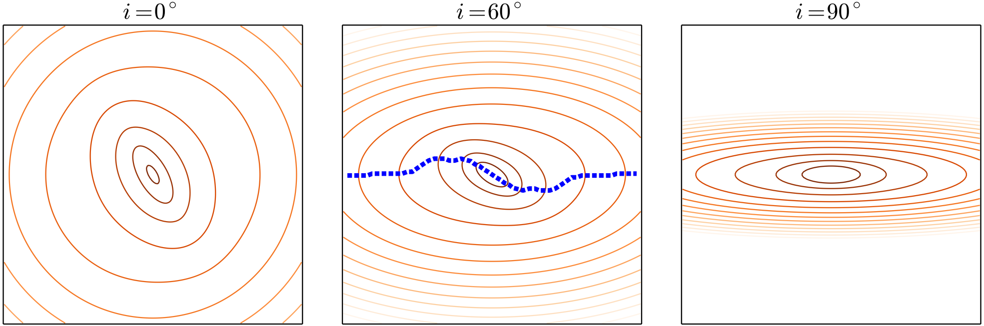




Isophote twists

When fitting the surface density by a superposition of several ellipsoidal components, one may allow the orientations of projected major axis to be different for each component. If these orientations change with radius, the 3d shape cannot be axisymmetric, but it may be triaxial with a radially varying intrinsic axis ratio, seen at an intermediate orientation (not aligned with any of its principal axes). The range of possible viewing angles is different for each component, and deprojection is possible only for their intersection.





Non-ellipsoidal isophotes

Often the isophotes are not quite ellipsoidal in shape.

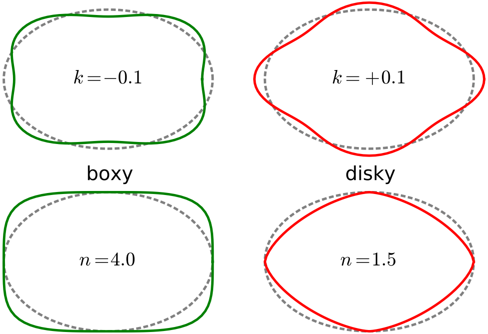
One may fit "boxy" or "disky" contours by adding a 4th harmonic to the radius of the ellipse,

$$X = a\ \cos\phi\,(1 + k\ \cos 4\phi),$$
$$Y = b\ \sin\phi\,(1 + k\ \cos 4\phi),$$

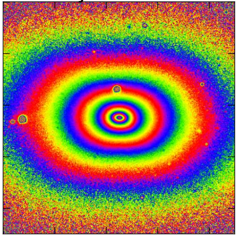
or by generalized ($n \neq 2$) ellipses specified by $\left(\frac{X}{a}\right)^n + \left(\frac{Y}{b}\right)^n = 1.$

Often the "boxy/peanut" or "X-shape" is associated with edge-on bars.

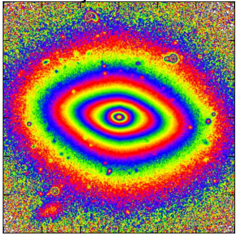
Note that such surface density profiles cannot be easily deprojected.



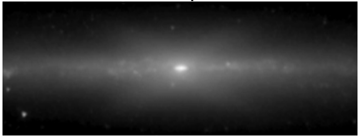
Boxy: NGC5322



Disky: NGC4660



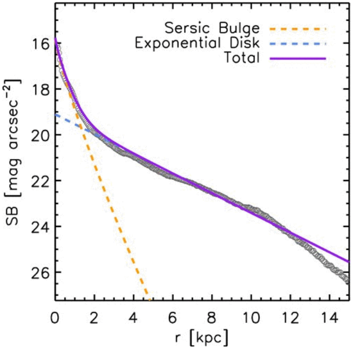
X-shaped: NGC3628



Deprojection in practice

Often the 2d image of a galaxy is fitted with a combination of several parametric profiles (e.g., an ellipsoidal Sérsic bulge plus an exponential disc), or a superposition of many ($\sim 5-10$) elliptical Gaussian components (Multi-Gaussian Expansion) [Emsellem+ 1994, Cappellari 2002].

Widely used codes: GALFIT [Peng+ 2002], IMFIT [Erwin 2015], MGEFIT [Cappellari 2002].



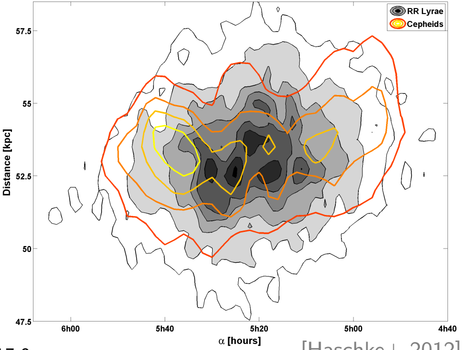
Caveats:

- Photometric decomposition into structural components (e.g., bulge+disc) may not adequately describe the actual physically distinct populations.
- Most observers are happy to stack multiple complicated profiles to best match the photometry, but the deprojection of the resulting surface density is next to impossible.
- Deprojection is formally unique for ellipsoidal models (even with isophote twists) for an assumed orientation, but the range of possible deprojections outside this class is usually ignored.

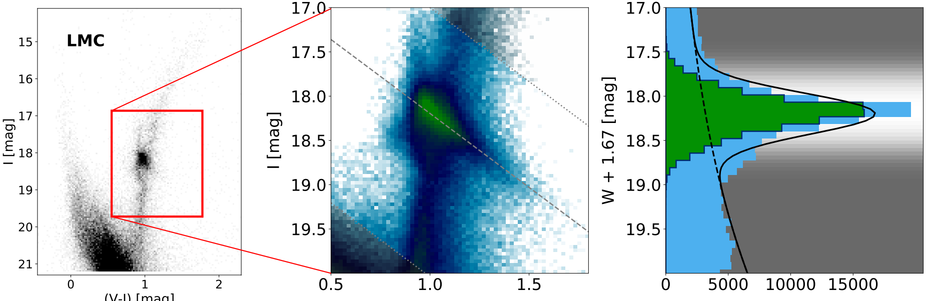
3d density from resolved stellar populations

Certain types of stars ("standard candles" such as RC, BHB, RR Lyrae or Cepheids) can be used to measure distance and reddening at different sightlines and construct maps of the mean distance and thickness (from their extinction-corrected apparent magnitudes).

Currently this is possible for the Milky Way and some satellites (Magellanic clouds, Sagittarius dSph).

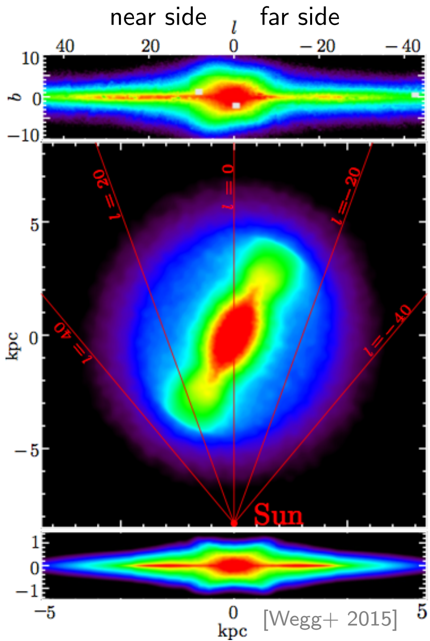


[Haschke+ 2012]

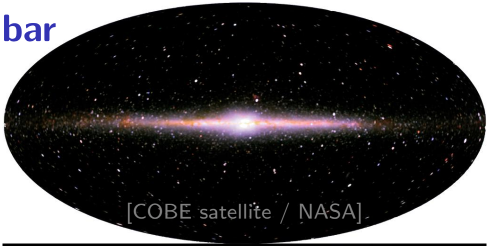


[Skowron+ 2021]

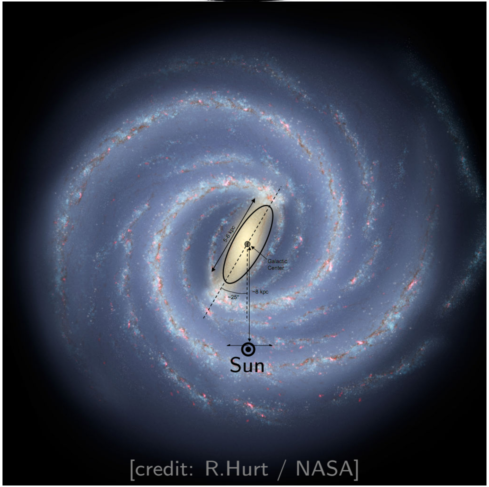
Perspective effects in the Galactic bar



[Wegg+ 2015]



[COBE satellite / NASA]



[credit: R.Hurt / NASA]

Summary

- Useful math concepts: fractional calculus, tomography in the Fourier space, rotation matrices.
- Deprojection of spherical profiles is unique (although ill-conditioned).
- Deprojection of axisymmetric profiles is non-unique, even if we specify the inclination angle.
- Ellipsoidal models *do* have a unique deprojection *for an assumed orientation* (in a limited range of angles, e.g. not too small inclinations in the axisymmetric case).
- Isophote twists require non-axisymmetric intrinsic shapes that vary with radius.