# Modern stellar dynamics, lecture 3: gravitational potential


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# Gravitational potential

$$\nabla^2\Phi = 4\pi\, G\, \rho \qquad \text{(in Newtonian gravity)}$$

$$G \approx 0.0043\, \frac{\mathrm{pc}\, (\mathrm{km/s})^2}{M_\odot}.$$

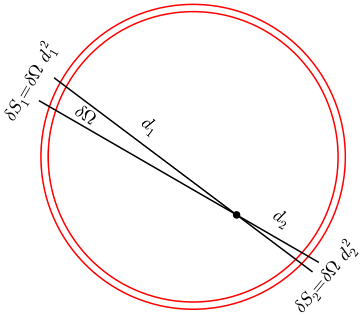
In galactic dynamics:

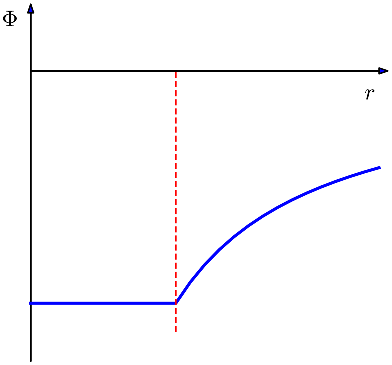
- neglect relativity;
- neglect cosmological expansion;
- [usually] $\Phi$ is negative and tends to zero at infinity.

Formal solution of the Poisson equation (*not* a practically useful expression):

$$\Phi(\mathbf{x}) = -\iiint d^3x'\, \frac{G\, \rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$$

# Gravitational potential of spherical systems




*Newton's 1st theorem*:
a body inside a spherical shell with uniform density experiences no net gravitational force.

*Newton's 2nd theorem*:
the force outside a spherical shell of total mass $M$ is the same as from a point mass.

Both results directly follow from Gauss's divergence theorem.

# Gravitational potential of spherical systems

density profile $\rho(r) = \frac{1}{4\pi\, G}\nabla^2\Phi = \frac{1}{4\pi\, G\; r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right)$

$\Downarrow$

enclosed mass $M(<r) \equiv \int_0^r \mathrm{d}s\; 4\pi\, s^2\, \rho(s)$

radial ~~force~~ acceleration $F_r(r) \equiv -\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r} = -\frac{G\, M(<r)}{r^2}$

circular velocity $v_{\mathrm{circ}}(r) \equiv \sqrt{r\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}}$

$\Downarrow$

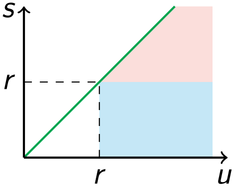
potential

$$\Phi(r) = \int_r^\infty \mathrm{d}u\; F_r(u) = -4\pi\, G \int_r^\infty \mathrm{d}u \int_0^u \mathrm{d}s\; \frac{s^2\,\rho(s)}{u^2}$$

$$= -4\pi\, G \int_0^\infty \mathrm{d}s\; s^2\,\rho(s) \int_{\max(r,s)}^\infty \mathrm{d}u\; \frac{1}{u^2}$$

$$= -4\pi\, G \left[\frac{1}{r}\int_0^r \mathrm{d}s\; s^2\,\rho(s) + \int_r^\infty \mathrm{d}s\; s\,\rho(s)\right]$$

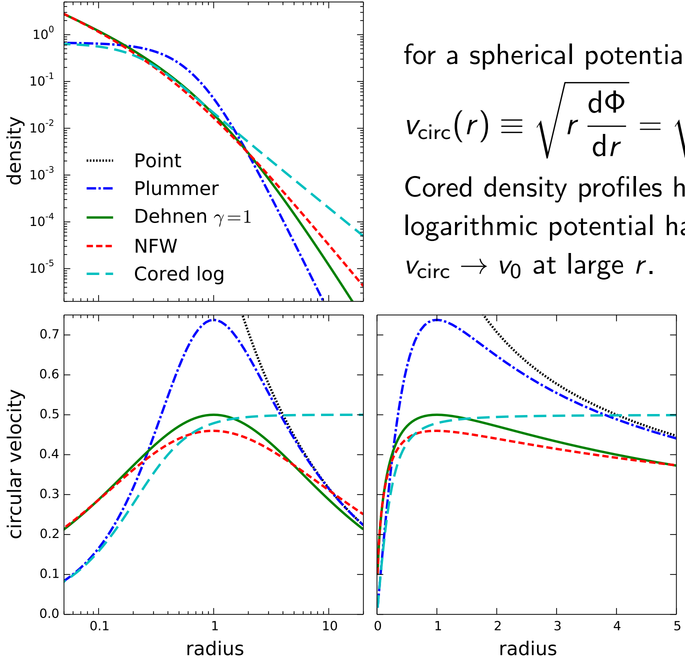
interior shells exterior shells



# Examples of spherical potentials

1. Power law: $\rho(r) = \rho_0 \left(r/a\right)^{-\gamma} \implies M(r) = \frac{4\pi\, \rho_0\, a^3}{3-\gamma}\left(r/a\right)^{3-\gamma}$,
   $$\Phi(r) = \begin{cases} \frac{4\pi\, G\, \rho_0\, a^2}{(3-\gamma)(2-\gamma)}\left(r/a\right)^{2-\gamma} & \text{if } \gamma \neq 2, \\ 4\pi\, G\, \rho_0\, a^2\, \ln\left(r/a\right) & \text{if } \gamma = 2. \end{cases}$$
2. Plummer model: $\Phi(r) = -\frac{G\, M}{\sqrt{r^2+a^2}}$.
3. Dehnen model: $\Phi(r) = -\frac{G\, M}{(2-\gamma)\, a}\left[1 - \left(\frac{r}{r+a}\right)^{2-\gamma}\right] \qquad (\gamma < 2)$.
4. NFW model: $\Phi(r) = -\frac{G\, M}{r} \ln\left[1 + \frac{r}{a}\right]$.
5. Cored logarithmic: $\Phi(r) = v_{\rm circ}^2 \ln\left(1 + r/a\right)$.

# Circular-velocity curves




for a spherical potential,

$$v_{\mathrm{circ}}(r) \equiv \sqrt{r\,\frac{\mathrm{d}\Phi}{\mathrm{d}r}} = \sqrt{\frac{G\,M(<r)}{r}}.$$

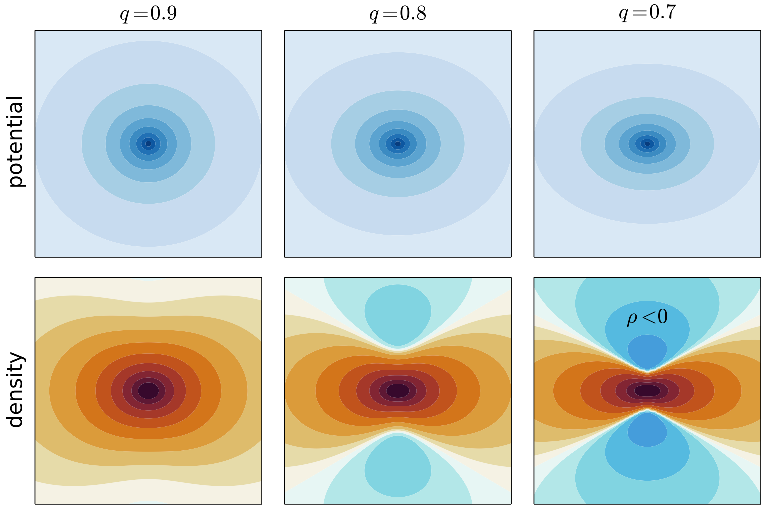
Cored density profiles have $v_{\mathrm{circ}} \propto r$ at small $r$;
logarithmic potential has a "flat rotation curve"
$v_{\mathrm{circ}} \to v_0$ at large $r$.

term strictly applicable only to gas motion, which is indeed nearly circular; mean stellar rotation velocity may be significantly smaller than $v_{\mathrm{circ}}$, so to avoid confusion, it's better to call this "circular-velocity curve"

# Ellipsoidal potentials

Substitute the ellipsoidal radius $m \equiv \sqrt{x^2 + (y/p)^2 + (z/q)^2}$ into expressions for spherical potential models (e.g., Dehnen $\gamma = 1$ in this example).

The result is disappointing – density becomes negative at large $r$.

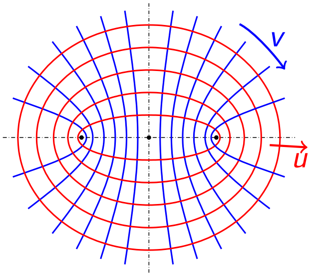


# Potential of ellipsoidal density profiles

Substitute the ellipsoidal radius $m \equiv \sqrt{x^2 + (y/p)^2 + (z/q)^2}$ into expressions for spherical *density* profiles and compute the potential.

Introduce a coordinate system based on *confocal* ellipses and hyperbolae:

not *concentric* (similar)

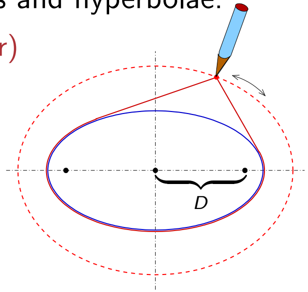



2d elliptic coordinates

ellipses: $$\frac{R^2}{\cosh^2 u} + \frac{z^2}{\sinh^2 u} = D^2$$

hyperbolae: $$\frac{R^2}{\sin^2 v} - \frac{z^2}{\cos^2 v} = D^2$$

$$R = D \cosh u \sin v,$$
$$z = D \sinh u \cos v,$$
$$u \geq 0,\ 0 \leq v \leq \pi$$




how to draw confocal ellipses

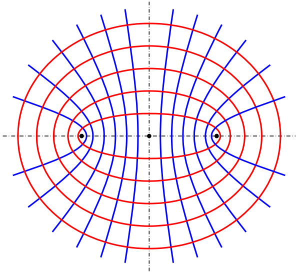
[Graves 1850, bishop of Limerick]

limiting cases:

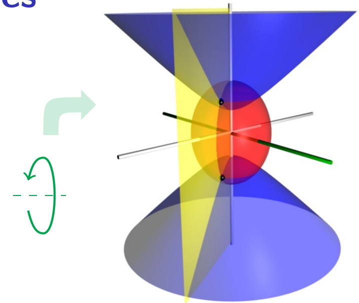
$D \to 0$ – spherical: $u \rightsquigarrow r,\ v \rightsquigarrow \theta$

$D \to \infty$ – cylindrical: $u \rightsquigarrow z,\ v \rightsquigarrow R$

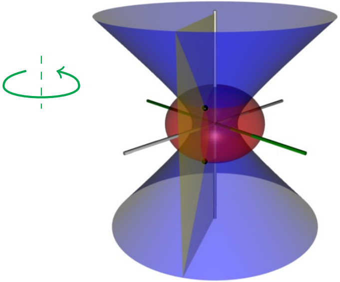
# Ellipsoidal coordinates



2d elliptic coordinates

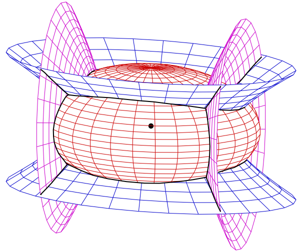


prolate spheroidal

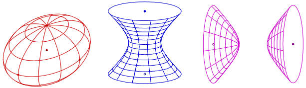


oblate spheroidal

[spheroid is an ellipsoid with two equal axes]



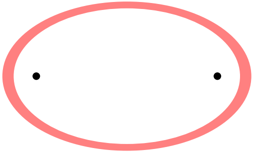
triaxial ellipsoidal



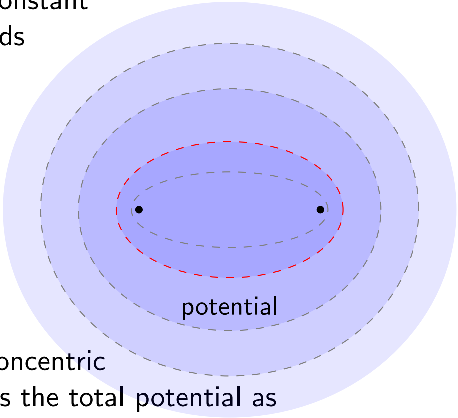
[from Wikipedia]

# Potential of ellipsoidal density profiles

It turns out that the potential created by a thin uniform-density homoeoid (region between two *similar* ellipsoidal shells) is constant inside the shell, and stratified on *confocal* ellipsoids outside the shell (i.e. gets rounder with radius).



density



potential

By breaking down a density profile stratified on concentric (similar) ellipsoids into thin shells, one can express the total potential as

$$\Phi(\mathbf{x}) = -2\pi\, G\, pq \int_0^\infty \mathrm{d}\tau \frac{\xi\big(m(\tau)\big)}{\sqrt{(\tau+1)(\tau+p^2)(\tau+q^2)}},$$

$$\xi(m) \equiv \int_m^\infty \mathrm{d}m\; m\,\rho(m), \quad m^2(\tau) \equiv \frac{x^2}{\tau+1} + \frac{y^2}{\tau+p^2} + \frac{z^2}{\tau+q^2}.$$

(typically evaluated numerically)

# Stäckel potentials

A general triaxial ellipsoidal coordinate system $\lambda, \mu, \nu$ is defined by two focal distances; coordinate lines of constant $\lambda$ are confocal ellipses, constant $\mu$ – one-sheet hyperboloids, and constant $\nu$ – two-sheet hyperboloids.

A potential in a triaxial ellipsoidal coordinate system has a Stäckel form if

$$\Phi(\lambda, \mu, \nu) = \frac{f_\lambda(\lambda)}{(\lambda-\mu)(\nu-\lambda)} + \frac{f_\mu(\mu)}{(\mu-\nu)(\lambda-\mu)} + \frac{f_\nu(\nu)}{(\nu-\lambda)(\mu-\nu)}.$$

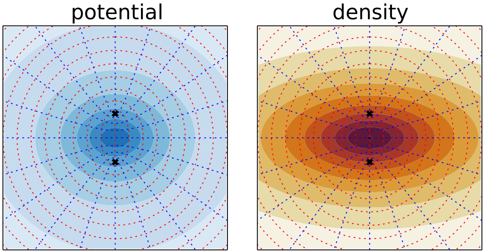
These potentials are important as the most general case in which the equations of motion are separable, as we will discuss in the next lecture.

A particularly "simple" example is the Perfect Ellipsoid model [de Zeeuw 1985]:

$$\rho(\mathbf{x}) = \frac{M\,a}{\pi^2\,p\,q}\,\frac{1}{\left[a^2 + x^2 + (y/p)^2 + (z/q)^2\right]^2};$$

($\Phi$ is expressed in terms of elliptic integrals).

Note that an *oblate* Perfect Ellipsoid density corresponds to a *prolate* spheroidal coordinate system for the potential!



# Multipole expansion

Laplacian in spherical coordinates $r, \theta, \phi$:

$$\nabla^2 \Phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta} \left( \sin\theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 \Phi}{\partial \phi^2}.$$

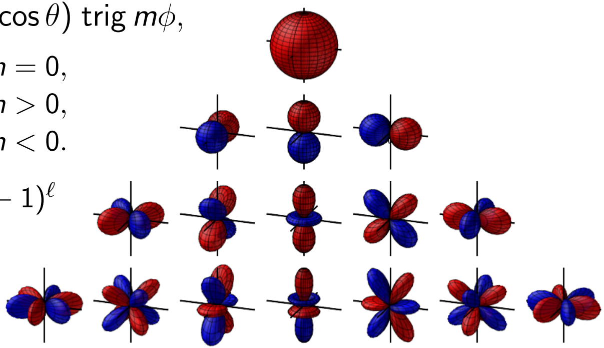
Consider first the angular part of the Laplacian – its eigenfunctions are [real-valued] spherical harmonics satisfying $\nabla^2 Y_\ell^m(\theta, \phi) = -\frac{\ell\,(\ell+1)}{r^2} Y_\ell^m(\theta, \phi)$:

$$Y_\ell^m(\theta, \phi) \equiv \sqrt{\frac{2\ell+1}{4\pi} \frac{(\ell-|m|)!}{(\ell+|m|)!}}\; P_\ell^{|m|}(\cos\theta)\, \mathrm{trig}\, m\phi,$$

$$\mathrm{trig}\, m\phi \equiv \begin{cases} 1 & \text{if } m = 0, \\ \sqrt{2}\cos m\phi & \text{if } m > 0, \\ \sqrt{2}\sin |m|\phi & \text{if } m < 0. \end{cases}$$

$$P_\ell^m(x) \equiv \frac{(-1)^m (1-x^2)^{m/2}}{2^\ell\, \ell!} \frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}} (x^2 - 1)^\ell$$

are associated Legendre functions, and $\ell \geq 0$, $|m| \leq \ell$ are integers.



# Multipole expansion

The functions $Y_\ell^m(\theta,\phi)$ form a complete orthonormal basis on the sphere:

$$\int_{-1}^{1} \mathrm{d}\,\cos\theta \int_0^{2\pi} \mathrm{d}\phi\; Y_\ell^m(\theta,\phi)\; Y_{\ell'}^{m'}(\theta,\phi) = \delta_\ell^{\ell'}\,\delta_m^{m'}.$$

Thus any smooth function $f(\theta,\phi)$ can be represented by an infinite series

$$f = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m}\, Y_\ell^m(\theta,\phi),$$

or approximated with any desired accuracy by a finite series (up to $\ell_{\mathrm{max}}$).

The coefficients of expansion are given by

$$f_{\ell m} = \int_{-1}^{1} \mathrm{d}\,\cos\theta \int_0^{2\pi} \mathrm{d}\phi\; f(\theta,\phi)\, Y_\ell^m(\theta,\phi).$$

Analogously to the Fourier series, the set of terms at a fixed $\ell$ describes the variation of the function on angular scales $\sim \pi/\ell$,

and is rotationally invariant: $\sum_{m=-\ell}^{\ell} f_{\ell m}^2$ is independent of the basis orientation.

# Multipole expansion of the potential

Now we can solve the Laplace equation $\nabla^2\Phi = 0$ by separation of variables:
assume $\Phi(r,\theta,\phi) = F(r)\,Y_\ell^m(\theta,\phi)$, then

$$\nabla^2\Phi = 0 = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\,\frac{\mathrm{d}F}{\mathrm{d}r}\right)Y_\ell^m - \frac{\ell\,(\ell+1)}{r^2}\,F\,Y_\ell^m.$$

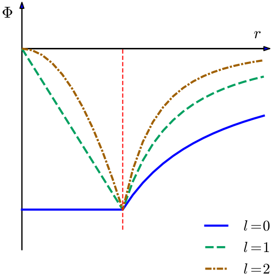
A power-law solution for the radial part is
$F^-(r) = r^\ell$ (at small radii) or $F^+(r) = r^{-\ell-1}$ (at large radii).
By joining the two solutions at a radius $s$ using Gauss's theorem,
$\frac{\partial\Phi_+}{\partial r}\Big|_{r=s} - \frac{\partial\Phi_-}{\partial r}\Big|_{r=s} = 4\pi\,G\,\Sigma$, we get the potential $\Phi_{\ell m}$ of a thin shell
with surface density $\Sigma(\theta,\phi) = \Sigma_{\ell m}\,Y_\ell^m(\theta,\phi)$:

$$\Phi_{\ell m}^- = -\frac{4\pi\,G\,s\,\Sigma_{\ell m}}{2\ell+1}\left(\frac{r}{s}\right)^\ell Y_\ell^m(\theta,\phi), \qquad \Phi_{\ell m}^+ = -\frac{4\pi\,G\,s\,\Sigma_{\ell m}}{2\ell+1}\left(\frac{r}{s}\right)^{-\ell-1} Y_\ell^m(\theta,\phi).$$

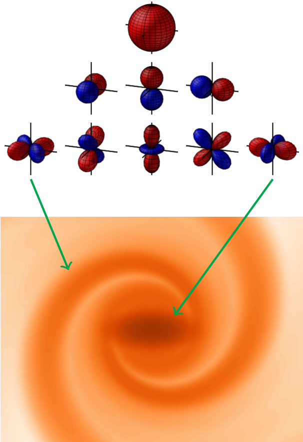
For a general density profile represented by spherical-harmonic coefficients $\rho_{\ell m}(r)$,

$$\Phi = -4\pi\,G\sum_{\ell,m}\frac{Y_\ell^m(\theta,\phi)}{2\ell+1}\left[\underbrace{r^{-\ell-1}\int_0^r \mathrm{d}s\;s^{\ell+2}\,\rho_{\ell m}(s)}_{\text{interior shells}} \;+\; \underbrace{r^\ell\int_r^\infty \mathrm{d}s\;s^{1-\ell}\,\rho_{\ell m}(s)}_{\text{exterior shells}}\right].$$

# Properties of multipole expansion




- The potential of a shell fades more quickly as one moves away in radius for higher-degree harmonics $\Rightarrow$ at large radii, the potential of any finite-mass model is close to a monopole.
- $\ell = 1$ terms describe the left/right (up/down, etc.) asymmetry and may be cancelled at any given radius by shifting the origin; however, it may not be possible to cancel them *everywhere* if the density profile is intrinsically lopsided.
- Triaxial systems aligned with the principal axes contain only terms with even $\ell$ and nonnegative even $m$; axisymmetric systems – only $m = 0$.
- $\ell = 2,\ m = 0$ term describes the flattening in the $z$ direction (in oblate systems, $\rho_{20} < 0$ and $\Phi_{20} > 0$).
- $\ell = 2,\ m = 2$ term measures the $y/x$ axis ratio ($\rho_{22} > 0$ if $x$ is the longer axis).
- Mirror-symmetric features such as spirals need both $m = 2$ (cosine) and $m = -2$ (sine) terms.



# Poisson equation in cylindrical coordinates

Laplacian in cylindrical coordinates $R, \phi, z$:

$$\nabla^2\Phi = \frac{1}{R}\frac{\partial}{\partial R}\left(R\,\frac{\partial\Phi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}.$$

Again use separation of variables: assume $\Phi(R, z, \phi) = F(R)\,G(\phi)\,H(z)$ and seek a solution to $\nabla^2\Phi = 0$ everywhere in half-space bounded by $z = 0$ (same strategy as in the spherical case, replacing shells by planes).

$$\frac{1}{R\,F(R)}\frac{\partial}{\partial R}\left(R\,\frac{\mathrm{d}F}{\mathrm{d}R}\right) + \frac{1}{R^2\,G(\phi)}\frac{\mathrm{d}^2 G}{\mathrm{d}\phi^2} \;=\; -\frac{1}{H(z)}\frac{\mathrm{d}^2 H}{\mathrm{d}z^2} \;=\; -k^2.$$

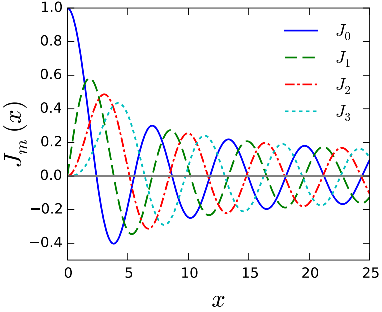
The solution for $H(z)$ is simple: $H(z) = \exp(\pm kz)$; we want it to decay both at $z \to +\infty$ and $z \to -\infty$ and have a break at $z = 0$, so $H(z) = \exp\big(-k|z|\big)$.

Multiplying the remaining expression by $R^2$, we again separate $R$ and $\phi$:

$$\frac{R}{F(R)}\frac{\partial}{\partial R}\left(R\,\frac{\mathrm{d}F}{\mathrm{d}R}\right) + k^2\,R^2 \;=\; -\frac{1}{G(\phi)}\frac{\mathrm{d}^2 G}{\mathrm{d}\phi^2} \;=\; m^2.$$

Thus the solutions for $G(\phi)$ are $\cos m\phi$, $\sin m\phi$.

# Poisson equation in cylindrical coordinates




We are left with a more complex equation for $F(R)$:

$$R\frac{\partial}{\partial R}\left(R\,\frac{\mathrm{d}F}{\mathrm{d}R}\right) + (k^2R^2 - m^2)\,F(R) = 0.$$

The solutions are Bessel functions $J_m(kR)$, $Y_m(kR)$ (the latter are singular at origin and hence not used).

$J_m(x)$ resemble sine functions multiplied by $1/\sqrt{x}$, and many of their properties are analogous to those of trigonometric functions.

In particular, they satisfy the following orthogonality relation:

$$\int_0^\infty \mathrm{d}R\ R\ J_m(kR)\ J_m(k'R) = \frac{\delta(k-k')}{k}.$$

The Hankel transform (a.k.a. Fourier–Bessel transform) is a direct analogue of the Fourier transform in cylindrical coordinates:

$$\widehat{F_m}(k) = \int_0^\infty \mathrm{d}R\ R\ J_m(kR)\ F(R) \quad\Longleftrightarrow\quad F(R) = \int_0^\infty \mathrm{d}k\ k\ J_m(kR)\ \widehat{F_m}(k).$$

# Poisson equation in cylindrical coordinates

A single solution of the Laplace equation outside the $z = 0$ plane is

$$\Phi_{km}(R, \phi, z) = J_m(kR) \operatorname{trig} m\phi \; \exp\left(-k|z|\right), \quad \operatorname{trig} m\phi \equiv \begin{cases} \cos m\phi & \text{if } m \geq 0, \\ \sin\, m\phi & \text{if } m < 0. \end{cases}$$

Using Gauss's theorem, we find that this potential is generated by the following surface density in the $z = 0$ plane:
$\Sigma_{km}(R, \phi) = -\frac{k}{2\pi\, G} J_m(kR) \operatorname{trig} m\phi$.

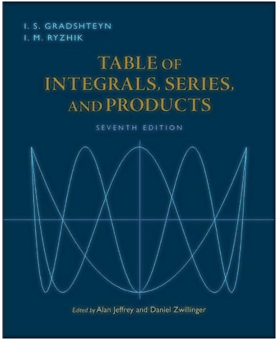
So the solution for an arbitrary surface density $\Sigma(R, \phi)$ is obtained by these steps:

1. Perform Fourier transform in $\phi$ to get $\Sigma_m(R) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\phi \operatorname{trig} m\phi \; \Sigma(R, \phi)$,
2. Perform Hankel transform in $R$ to get $\widehat{\Sigma_m}(k) = \int_0^\infty \mathrm{d}R \; R \; J_m(kR) \; \Sigma_m(R)$.
3. Corresponding term in the potential: $\widehat{\Phi_m}(k, z) = -\frac{2\pi\, G}{k} \widehat{\Sigma_m}(k) \; \exp\left(-k|z|\right)$.
4. The entire potential is given by the inverse Hankel and Fourier transforms:
   $\Phi_m(R, z) \; = \int_0^\infty \mathrm{d}k \; k \; J_m(kR) \; \widehat{\Phi_m}(k, z)$,
   $\Phi(R, \phi, z) = \sum_{m=-\infty}^{\infty} \Phi_m(R, z) \operatorname{trig} m\phi$.

# Example: potential of an axisymmetric exponential disc

In simple cases, some of the above steps may be performed analytically.
Consider an infinitely thin disc with $\Sigma(R) = \frac{M}{2\pi\, a^2} \exp\left(-R/a\right)$.
Its azimuthal Fourier transform obviosly consists of a single $m = 0$ term,
and the subsequent Hankel transform gives [GR 6.623]

$$\widehat{\Sigma_0}(k) = \frac{M}{2\pi\, a^2} \int_0^\infty \mathrm{d}R\; R\; J_0(kR)\; \exp(-R/a) = \frac{M}{2\pi\,(k^2 a^2+1)^{3/2}}.$$




Now the potential is

$$\Phi_0(R,z) = -2\pi G \int_0^\infty \mathrm{d}k\; J_0(kR)\; \exp\left(-k|z|\right)\; \widehat{\Sigma_0}(k).$$

Unfortunately, even Gradshteyn & Ryzhik cannot help with computing this integral analytically, except for $z = 0$:

$$\Phi_0(R,0) = -\frac{G\,M\,R}{2a^2}\left[I_0\left(\frac{R}{2a}\right) K_1\left(\frac{R}{2a}\right) - K_0\left(\frac{R}{2a}\right) I_1\left(\frac{R}{2a}\right)\right].$$

Consider now a finite-thickness disc with a separable profile: $\rho(R,z) = \Sigma(R)\, h(z)$.
By breaking it up into thin planes at each $z'$, we get $\Phi(R,z) = \int_{-\infty}^{\infty} \mathrm{d}z'\; \Phi_0(R,z')\, h(z-z')$
and fortunately, the integral $\int_{-\infty}^{\infty} \mathrm{d}z'\; h(z-z')\; \exp(-k|z-z'|)$ can be computed analytically for exponential or isothermal $h(z)$, still leaving a 1d numerical integral in $k$.
This is not too exciting, and an alternative multipole-based technique is more convenient.

# Multipole potential for separable axisymmetric density profiles

Let $\rho(R,z) = \Sigma(R)\,h(z)$ and $H(z)$ defined as $H''(z) = h(z)$.

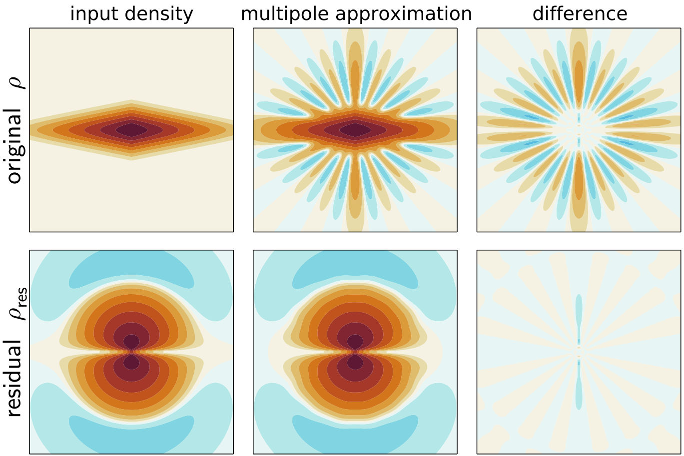
Represent the potential as $\Phi(R,z) = 4\pi\,G\,\Sigma(r)\,H(z) + \Phi_{\rm res}(R,z)$,

where $\Phi_{\rm res}$ is generated by the "residual density"

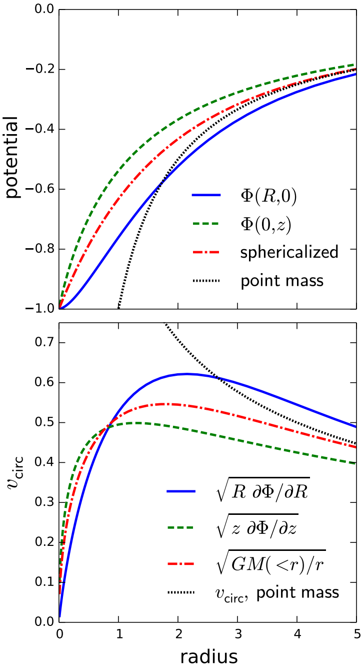
$$\rho_{\rm res} = \rho(R,z) - \big[\Sigma(R) - \Sigma(r)\big]\,h(z) - \tfrac{2}{r}\,\Sigma'(r)\,\big[H(z) + z\,H'(z)\big].$$

This density is not strongly concentrated towards the disc plane and can be efficiently represented by a spherical-harmonic expansion, unlike the original density [Kuijken&Dubinski 1994].

Example for a radially exponential, vertically isothermal disc with $\ell_{\max} = 16$.



# Circular-velocity curves in flattened potentials




Define $v_{\rm circ} \equiv \sqrt{R\,\dfrac{\partial\Phi}{\partial R}}$,

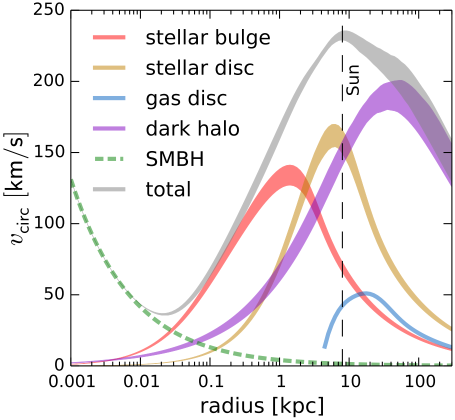
note that $v_{\rm circ} \neq \sqrt{\dfrac{G\,M(<r)}{r}}$:

the potential grows slower along the major axis, and $v_{\rm circ}$ eventually reaches higher values than in a spherical system with the same mass profile.

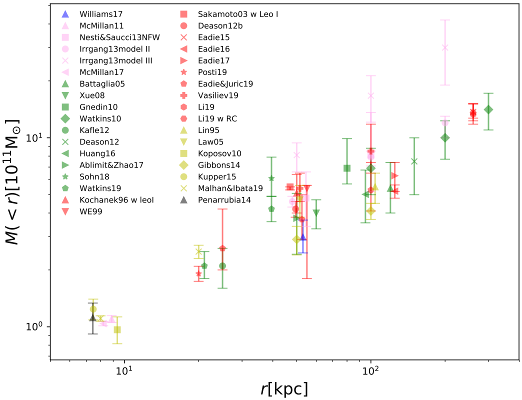
The deviation between $v_{\rm circ}$ and $\sqrt{G\,M(<r)/r}$ is largest when the density is very flattened, and gradually decreases for thicker discs (the illustration shows an infinitely thin exponential disc with $a = 1$).

# Potential of the Milky Way

- Circular velocity peaks around the Solar radius (8.2 kpc) at $\sim 235 \pm 5$ km/s
- Stars and dark halo have roughly equal contribution at this radius
- Mass profile at large radii ($\gtrsim 20$ kpc) is still rather uncertain
- Total ("virial") mass within 250–300 kpc is likely $\sim (0.8 - 1.5) \times 10^{12}\ M_\odot$




[McMillan 2017] – pre-Gaia




[Wang+ 2020]

# Summary

- Useful math concepts: confocal ellipsoids, Fourier, Hankel and spherical-harmonic transforms.
- Equipotential surfaces are rounder than equidensity surfaces.
- Potential of spherical systems is easy to compute (1d integration).
- Non-spherical systems generally require 3d integration
  (with some exceptions, e.g., in ellipsoidally-stratified density profiles);
- Multipole expansion is often the most efficient way of [approximately] computing the potential, but in the original form it is inaccurate for disky systems.
- In practice, the solution of the Poisson equation is a solved problem: there exist efficient codes for computing the potential numerically for an arbitrary density profile.