Modern stellar dynamics, lecture 4: motion of stars

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Hamiltonian mechanics

Fundamental concepts of Hamiltonian mechanics:

Hamiltonian $H(\mathbf{q}, \mathbf{p})$,
where $\mathbf{q}$ are generalized coordinates, $\mathbf{p}$ are generalized momenta.

Equations of motion:

$$\begin{aligned} \dot{\mathbf{q}} &= \partial H/\partial \mathbf{p}, \\ \dot{\mathbf{p}} &= -\partial H/\partial \mathbf{q}. \end{aligned}$$

In the simplest case, $\mathbf{q}$ would be Cartesian coordinates $\mathbf{x} \equiv \{x, y, z\}$,
and $\mathbf{p}$ – corresponding velocity components $\mathbf{v} \equiv \{v_x, v_y, v_z\}$.

$$H(\mathbf{x}, \mathbf{v}) = \Phi(\mathbf{x}) + \tfrac{1}{2}|\mathbf{v}|^2.$$

1d harmonic potential

Simplest possible Hamiltonian system with bound motion:

$$\text{harmonic oscillator:} \quad H(x,p) = \tfrac{1}{2}\Omega^2 x^2 + \tfrac{1}{2}p^2.$$

Not a totally idealized case:

$$\Phi(x) \propto x^2 \implies \rho(x) \propto \frac{d^2\Phi}{dx^2} \propto \text{const.}$$

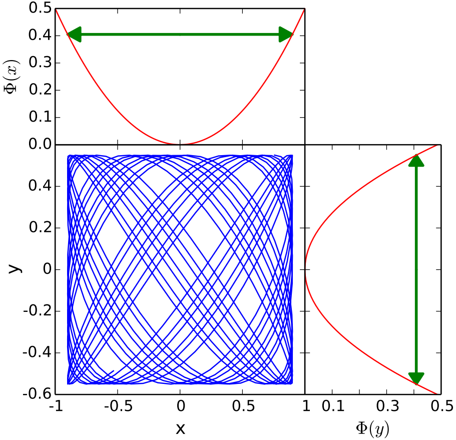
Constant-density cores are commonly encountered in many stellar systems.

2d harmonic potential

Same in more than one dimension:

$$H(\mathbf{x},\mathbf{p}) = \frac{1}{2}\left[\Omega_x^2\,x^2 + p_x^2\right] + \frac{1}{2}\left[\Omega_y^2\,y^2 + p_y^2\right] + \dots$$

Separable Hamiltonian – each dimension can be integrated independently; the orbit is a Lissajous figure filling a rectangle (if frequencies are incommensurable).



2d planar motion in a spherical potential

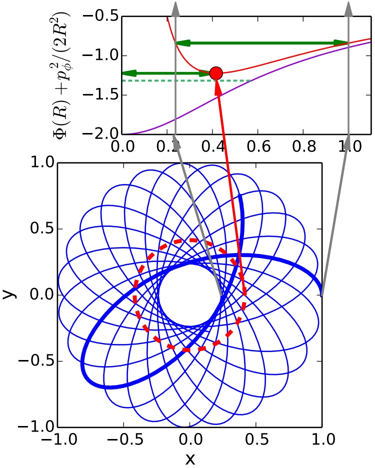
2d Hamiltonian in polar coordinates R, ϕ for a spherically-symmetric potential:

$$H(\mathbf{x}, \mathbf{p}) = \Phi(R) + \tfrac{1}{2}\left[p_R^2 + p_\phi^2/R^2\right].$$

Since H is independent of ϕ, the Hamilton equation reads
$\dot{p}_R = -\partial H/\partial\phi = 0$,
and the corresponding momentum p_ϕ
(angular momentum $L \equiv R\, v_\phi = R^2\,\dot{\phi}$)
is an integral of motion.

The remaining 1d motion occurs in the effective potential

$$\Phi_{\text{eff}}(R) \equiv \Phi(R) + \frac{L^2}{2\,R^2}.$$



2d planar motion in a spherical potential

The radial momentum ($\equiv$ velocity) is $p_R(R) = \pm\sqrt{2\big[E - \Phi(R)\big] - L^2/R^2}$;

bound motion occurs between $R_{\rm peri}$ and $R_{\rm apo}$ – the two roots of $p_R(R) = 0$ (peri- and apocentre radii). The period of radial oscillations is

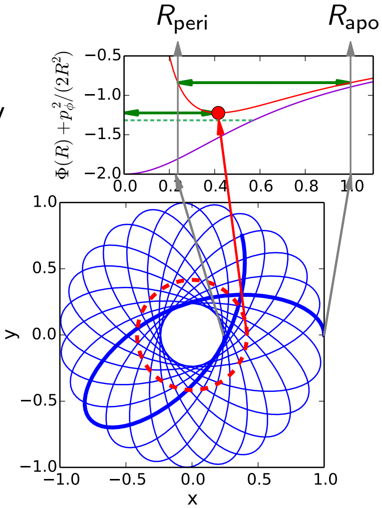
$$T_R \equiv 2\int_{R_{\rm peri}}^{R_{\rm apo}} \frac{dR}{p_R(R)}.$$

During this time, the azimuthal angle increases by

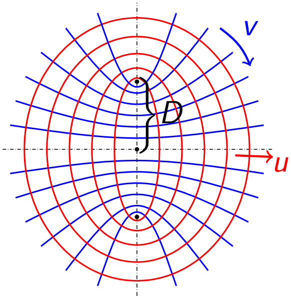
$$\Delta\phi = \int_0^{T_R} dt\ \dot\phi = 2\int_{R_{\rm peri}}^{R_{\rm apo}} \frac{dR}{p_R(R)}\frac{L}{R^2}.$$

The azimuthal period is defined as $T_\phi \equiv \dfrac{2\pi}{\Delta\phi}\ T_R$.

In general, the two periods are not commensurable, hence the orbit densely fills an annulus.
For realistic potentials, $1/2 \le T_R/T_\phi \le 1$, with the two limiting cases attained respectively in the harmonic and Kepler potentials.



Orbits in 2d Stäckel potentials



Consider the 2d elliptical coordinate system u, v

[note the different orientation w.r.t the previous lecture!]

$$\begin{aligned} x &= D \sinh u \ \ \sin v \\ y &= D \cosh u \ \cos v \end{aligned}, \qquad 0 \le u, \ \ 0 \le v \le 2\pi$$

The generalized momenta corresponding to u, v are

$$p_u = D^2 \left(\sinh^2 u + \sin^2 v\right) \dot{u},$$
$$p_v = D^2 \left(\sinh^2 u + \sin^2 v\right) \dot{v},$$

and the Hamiltonian in these coordinates is

$$H = \Phi(u, v) + \frac{p_u^2 + p_v^2}{2D^2 \left(\sinh^2 u + \sin^2 v\right)}.$$

If the potential has the form $\Phi(u, v) = \dfrac{U(u) - V(v)}{\sinh^2 u + \sin^2 v}$, then we may rearrange the above expression (replacing H by the energy E) to get

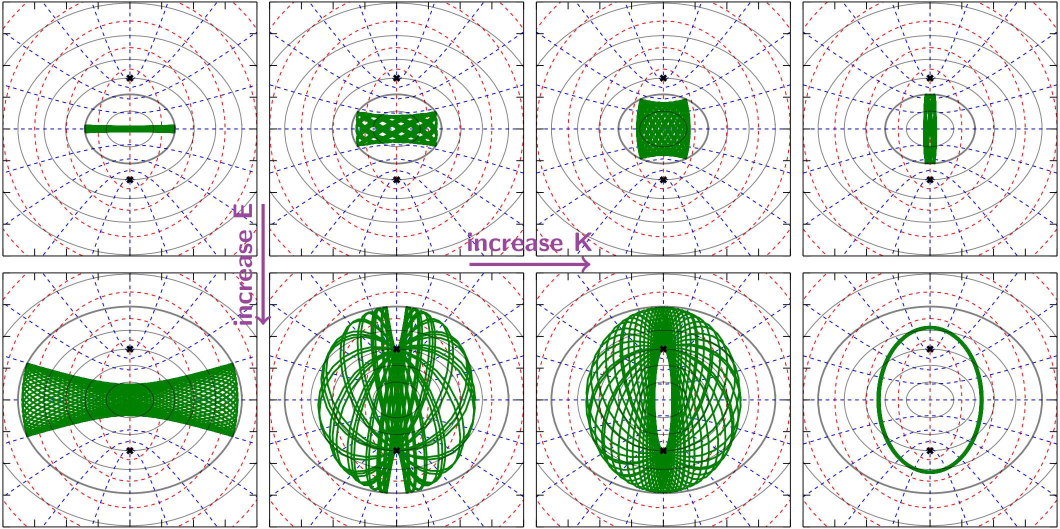
$$2D^2 \left[E \sinh^2 u - U(u)\right] - p_u^2 \ = \ p_v^2 - 2D^2 \left[E \sin^2 v + V(v)\right].$$

Since the lhs depends only on u and rhs – only on v, both must equal to some constant K. Thus the motion in both coordinates is separable.

Orbits in 2d Stäckel potentials

Example: the *oblate* Perfect Ellipsoid potential in the *prolate* elliptic coordinates.

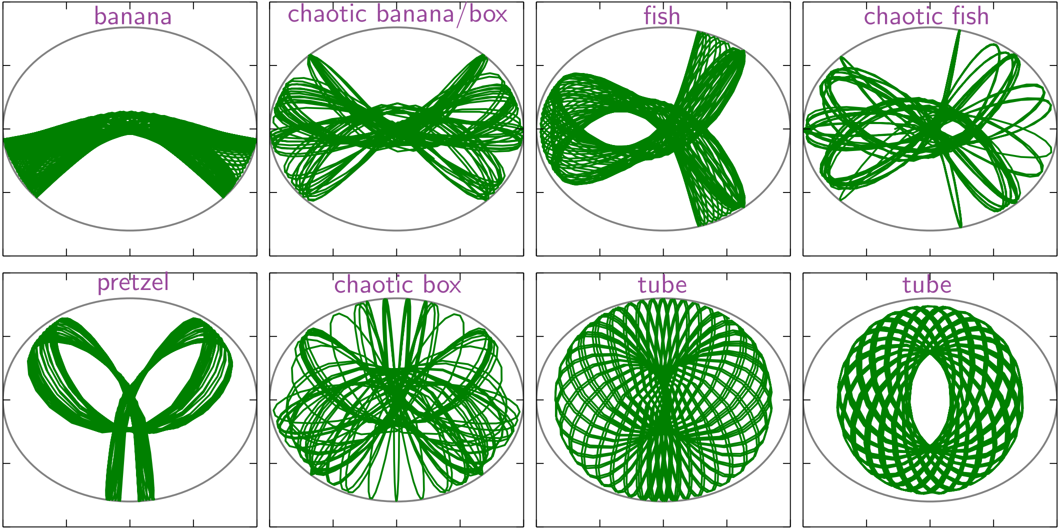
The equipotential surfaces (shown by black) are oblate (though *not* ellipsoidal), thus for low enough energies they do not enclose the focal points, and all orbits are boxes. For energies above the critical one, there is a transition between boxes and loops.



Orbits in generic 2d potentials

Example: the singular logarithmic potential.

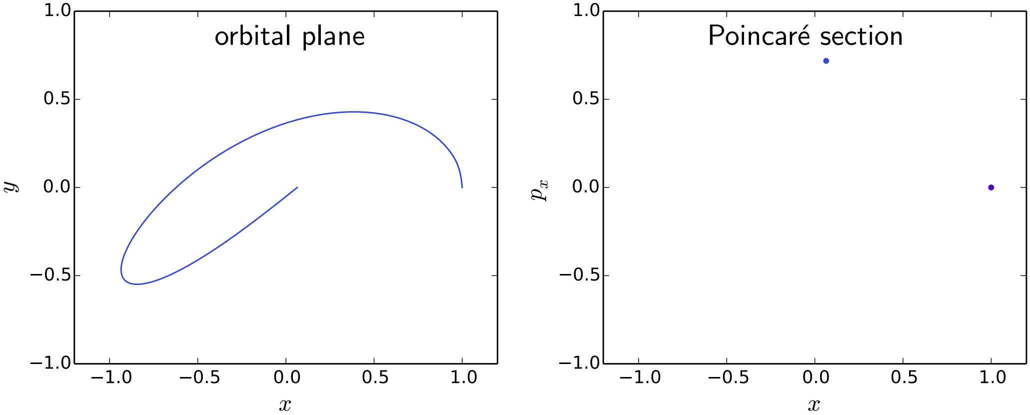
Non-axisymmetric potentials usually have box and loop orbits, but in addition may support various high-order resonant orbits, as well as chaotic ones. Singular potentials do not have pure boxes (any orbit passing too close to the centre is ridden with chaos).



Poincaré surface of section

A convenient tool for analyzing orbits in 2d Hamiltonian systems at a fixed E (e.g., motion in the equatorial plane, or in the meridional plane of an axisymmetric potential at a fixed L_z)

1. Numerically integrate the trajectory: $x(t), y(t), p_x(t), p_y(t)$.
2. Every time it passes through the axis $y = 0$ with $\dot{y} > 0$, put a point on the x, p_x plane.



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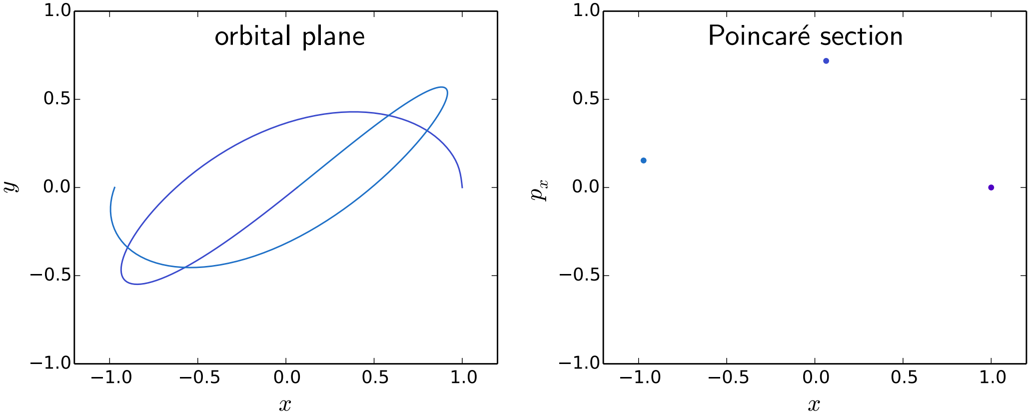

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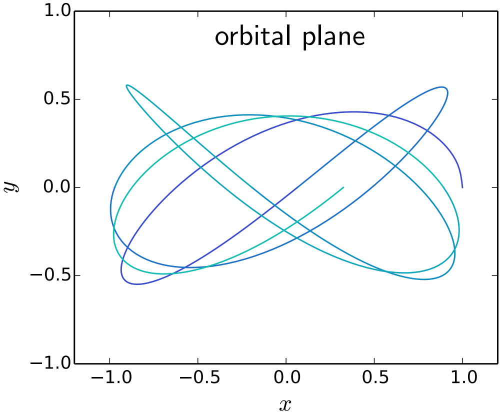

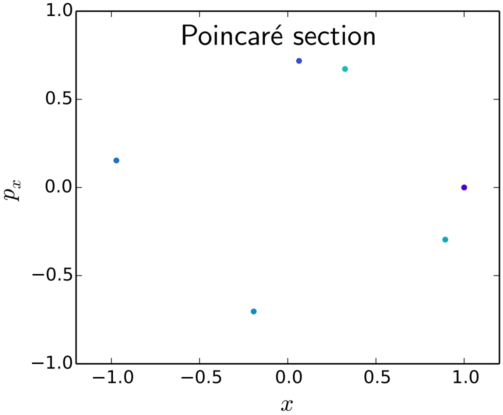

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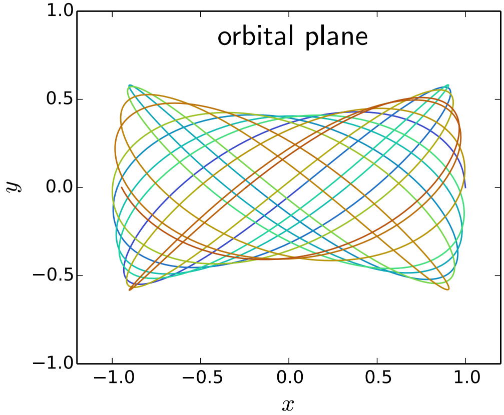

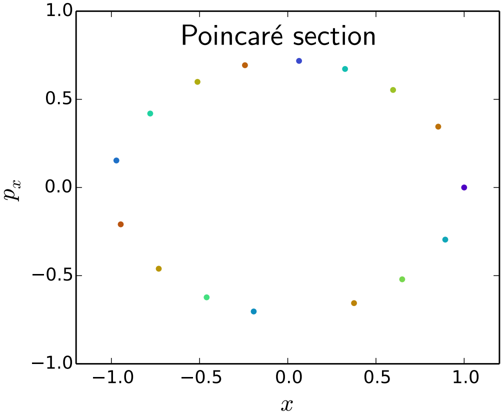

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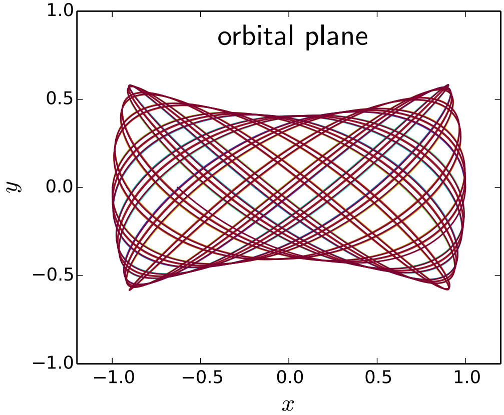
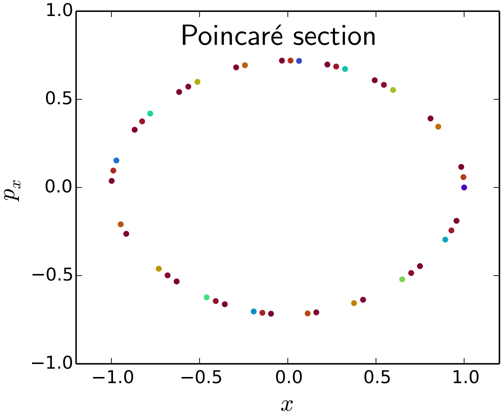
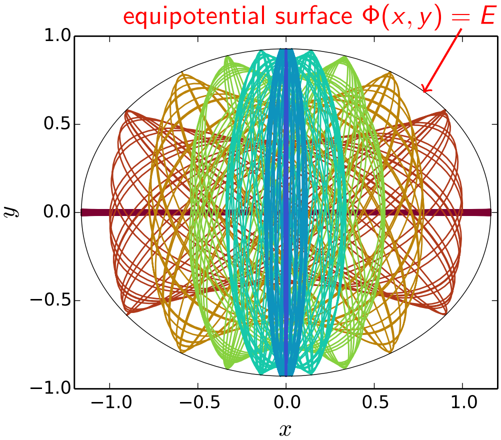


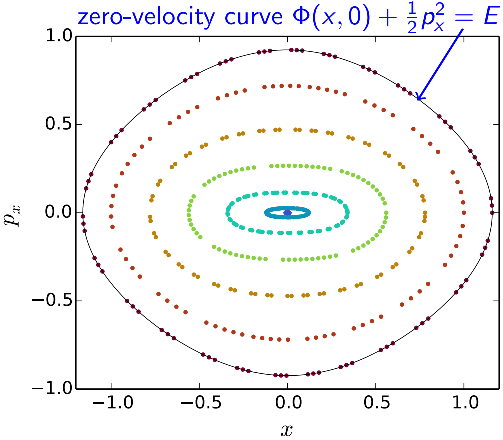


Poincaré surface of section

A convenient tool for analyzing orbits in 2d Hamiltonian systems at a fixed E
(e.g., motion in the equatorial plane, or in the meridional plane of an axisymmetric potential at a fixed L_z)

3. Each orbit corresponds to a closed loop in this plane.
4. Repeat for many different initial conditions to get the "phase portrait" of the Hamiltonian.





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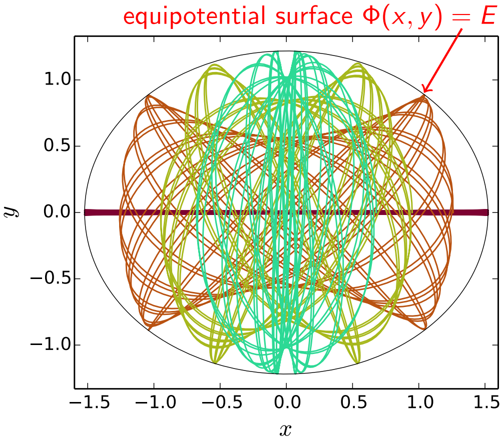
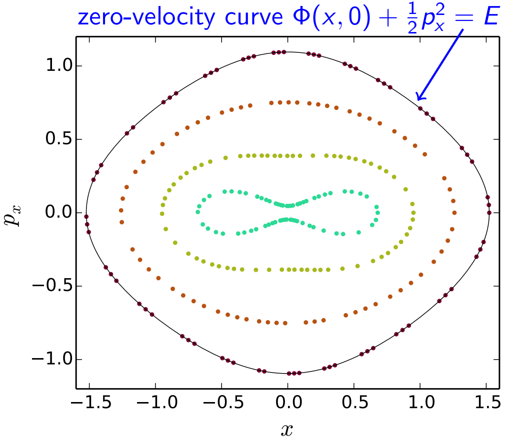
5. Now repeat this exercise for a different choice of energy E.





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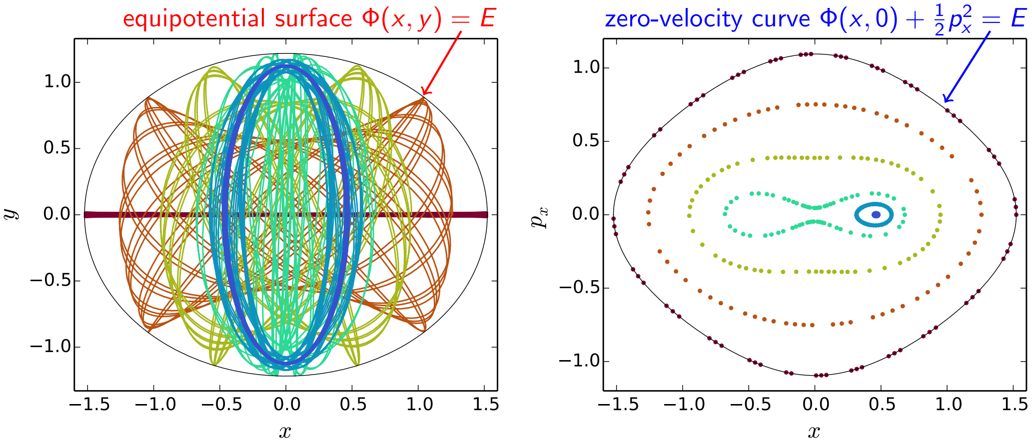
5. Now repeat this exercise for a different choice of energy E.

6. This portrait may contain more than one orbit family!



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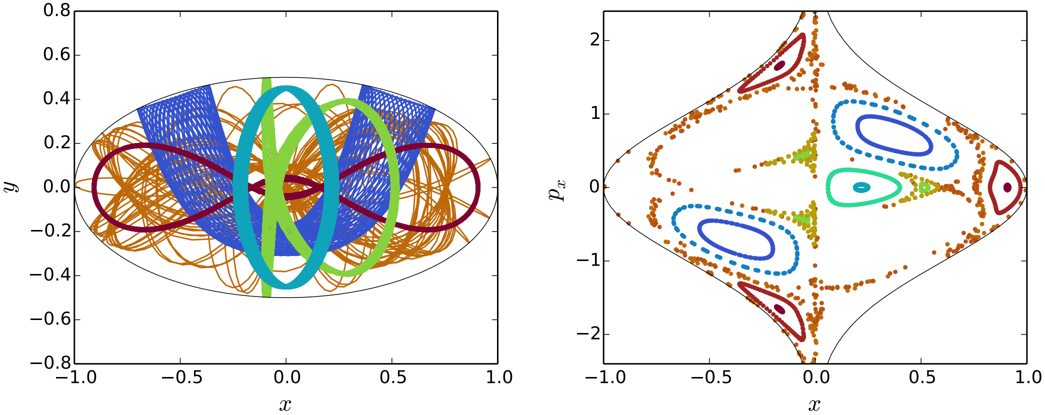
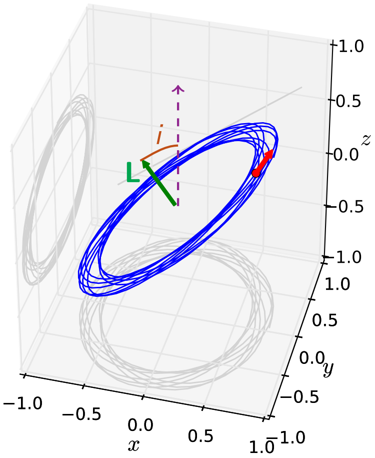
A convenient tool for analyzing orbits in 2d Hamiltonian systems at a fixed E (e.g., motion in the equatorial plane, or in the meridional plane of an axisymmetric potential at a fixed L_z)

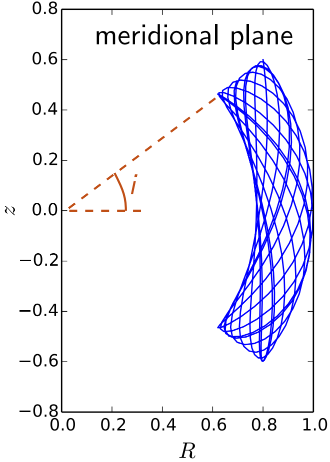
7. High-order resonances appear as orderly layered contour sets in the Poincaré surface, and chaotic orbits – as scattered layers separating the resonant islands.



Motion in spherical 3d potentials

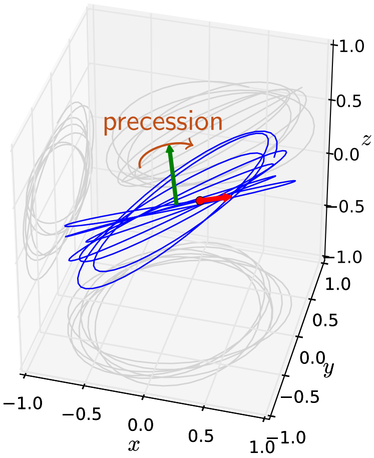
Since all components of angular momentum vector **L** are conserved, the orbit lies in a single plane, but it needs not be the equatorial $(x - y)$ plane. The inclination angle is defined as $i = \arccos(L_z/L)$, and the orbit in the meridional plane $(R - z)$ sweeps a section of an annulus.

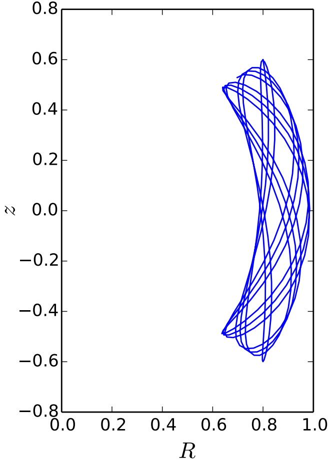




Motion in axisymmetric 3d potentials

In the axisymmetric case, the orbital plane still keeps a roughly constant inclination for most orbits, but additionally precesses about the z axis.



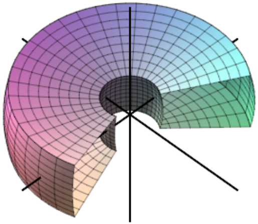


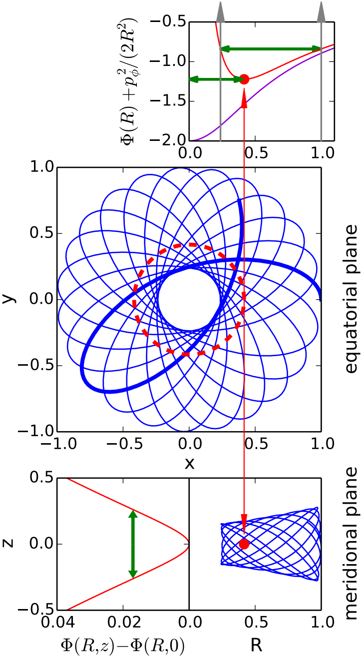
Motion in axisymmetric 3d potentials

3d Hamiltonian in cylindrical coordinates
$\mathbf{x} \equiv \{R, \phi, z\}$, $\mathbf{p} \equiv \{\dot{R}, R^2\dot{\phi}, \dot{z}\}$:
$H(\mathbf{x}, \mathbf{p}) = \Phi(R, z) + \frac{1}{2}\left[p_R^2 + p_\phi^2/R^2 + p_z^2\right]$.

Again $p_\phi \equiv R^2\dot{\phi}$ is the conserved z-component of angular momentum L_z, motion in ϕ separates out, and motion in the 2d meridional plane is governed by a 2d effective potential $\Phi_{\text{eff}}(R, z) = \Phi(R, z) + \dfrac{L_z^2}{2\,R^2}$.

The most common z-axis tube orbit looks like a "rectangular torus", resembling a box in meridional cross-section.

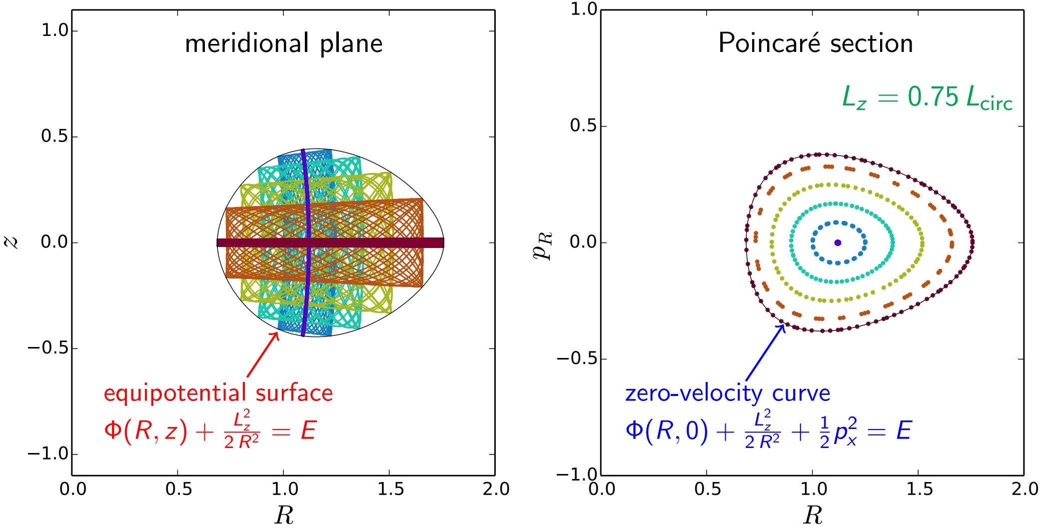




Poincaré section in axisymmetric 3d potentials

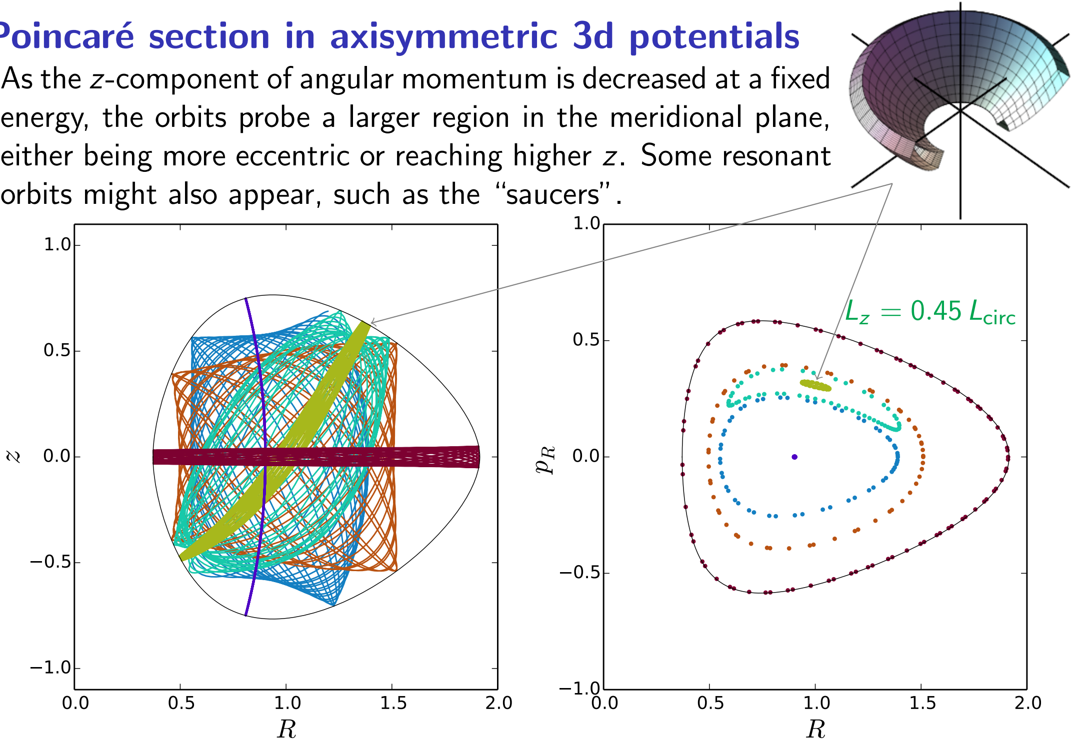
For any value of E, the value of L_z can be between 0 and the maximum possible value $L_{\rm circ}(E)$ – the angular momentum of a circular orbit in the $z=0$ plane.

For $L_z/L_{\rm circ}(E) \to 1$, all orbits are tubes, stay close to the equatorial plane and cover a small radial range.



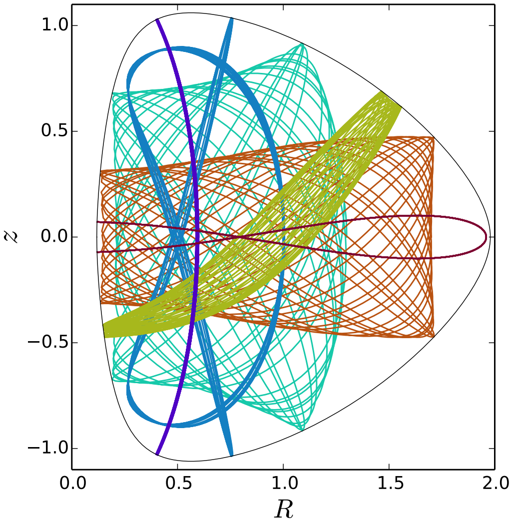
Poincaré section in axisymmetric 3d potentials

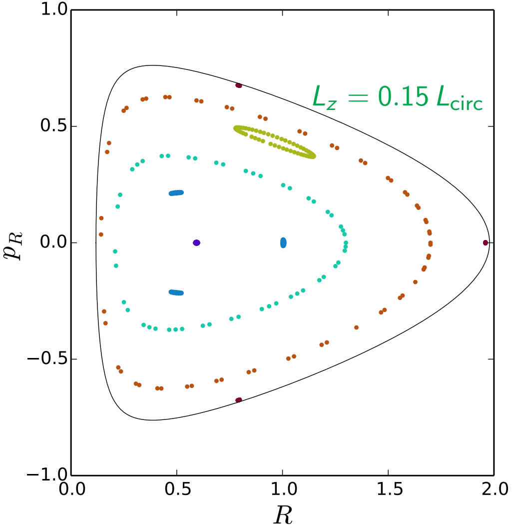
As the z-component of angular momentum is decreased at a fixed energy, the orbits probe a larger region in the meridional plane, either being more eccentric or reaching higher z. Some resonant orbits might also appear, such as the "saucers".



Poincaré section in axisymmetric 3d potentials

At low L_z, even more bizarre types of resonant orbits spring into existence. The $L_z = 0$ case is identical to the motion in a non-spherical 2d potential, this time in the $x - z$ rather than $x - y$ plane (extending to "negative R").





Epicyclic motion

For close-to-circular orbits near the disc plane, the motion in R and z is nearly separable and close to harmonic.
The guiding centre radius R_g is the minimum of the effective potential $\Phi_\mathrm{eff} \equiv \Phi + L_z^2/(2R^2)$, and

$$\Phi_\mathrm{eff}(R,z) \approx \Phi_\mathrm{eff}(R_\mathrm{g},0) + \frac{(R-R_\mathrm{g})^2}{2}\frac{\partial^2\Phi_\mathrm{eff}}{\partial R^2} + \frac{z^2}{2}\frac{\partial^2\Phi}{\partial z^2}.$$

Epicyclic frequencies at R_g:

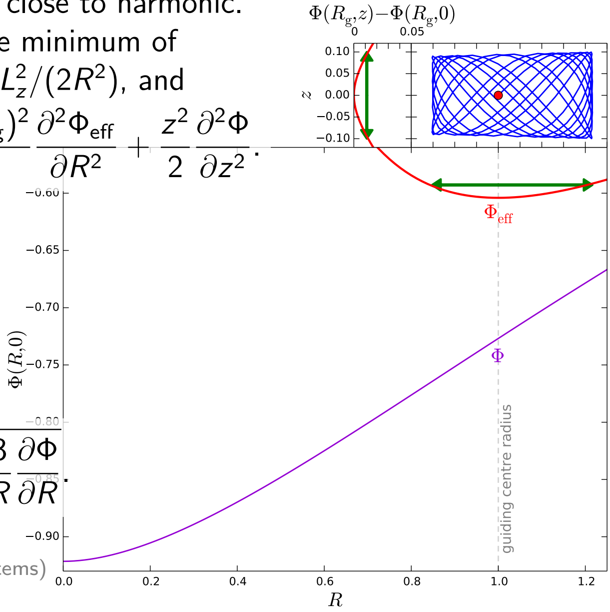
azimuthal $\Omega \equiv \dfrac{L_z}{R^2} = \sqrt{\dfrac{1}{R}\dfrac{\partial\Phi}{\partial R}}.$

vertical $\nu \equiv \sqrt{\dfrac{\partial^2\Phi}{\partial z^2}}.$

radial $\kappa \equiv \sqrt{\dfrac{\partial^2\Phi_\mathrm{eff}}{\partial R^2}} = \sqrt{\dfrac{\partial^2\Phi}{\partial R^2} + \dfrac{3}{R}\dfrac{\partial\Phi}{\partial R}}.$

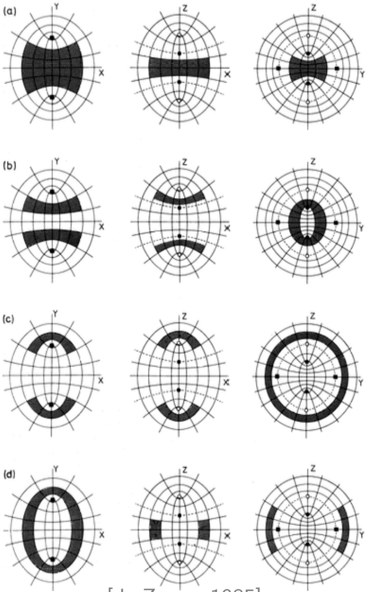
Typically $\Omega \le \kappa \le 2\Omega,\;\; \nu \ge \Omega.$

(for oblate systems)

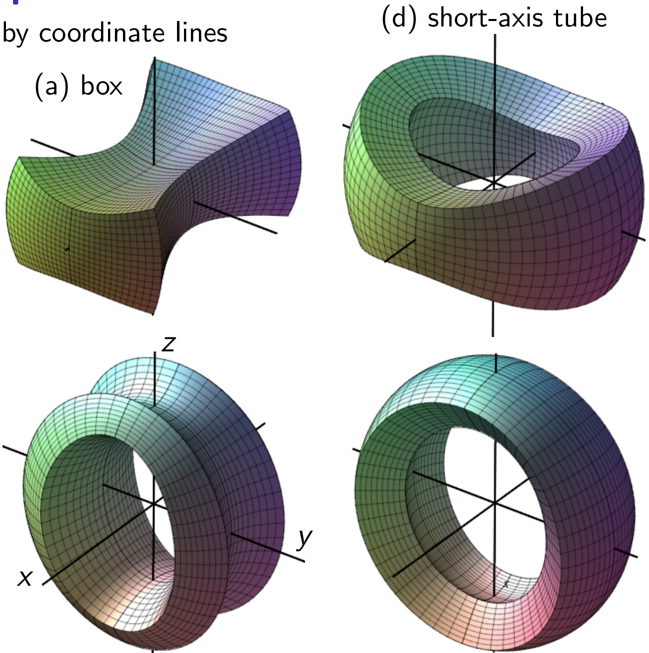


Orbits in triaxial Stäckel potentials

4 main classes of orbits bounded by coordinate lines



[de Zeeuw 1985]



(b) inner long-axis tube

(c) outer long-axis tube

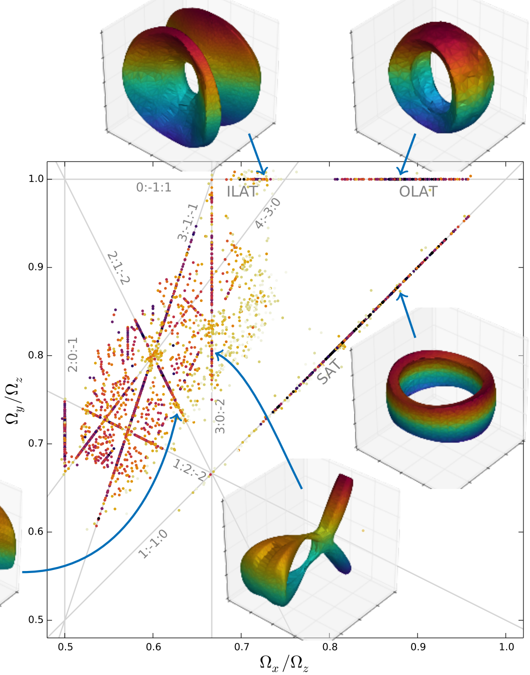
Orbits in generic triaxial potentials, frequency maps

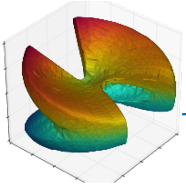
Most potentials support the same 4 main classes of orbits, but in addition may contain various resonant orbit families, as well as chaotic orbits.

All orbits with the given E can be represented on a frequency map

[Papaphilippou&Laskar 1998, Valluri&Merritt 1998]

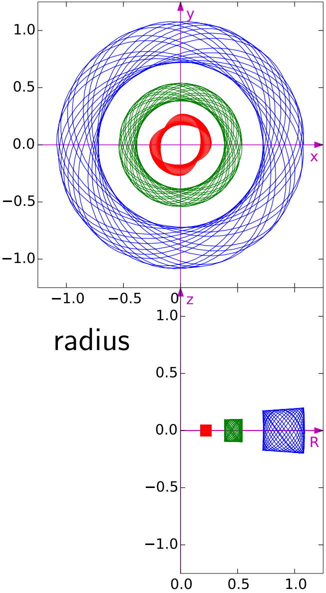
as ratios of frequencies Ω_x/Ω_z, Ω_y/Ω_z, where they concentrate along lines representing stable resonances: $k_x\Omega_x + k_y\Omega_y + k_z\Omega_z = 0$ with integer k.





Orbits and integrals of motion

How large is the variety of orbits?



radius

Orbits and integrals of motion

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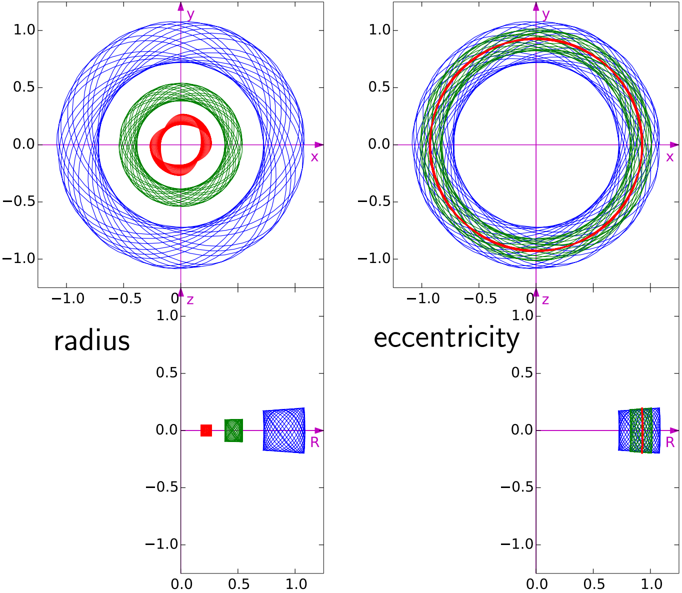

Orbits and integrals of motion

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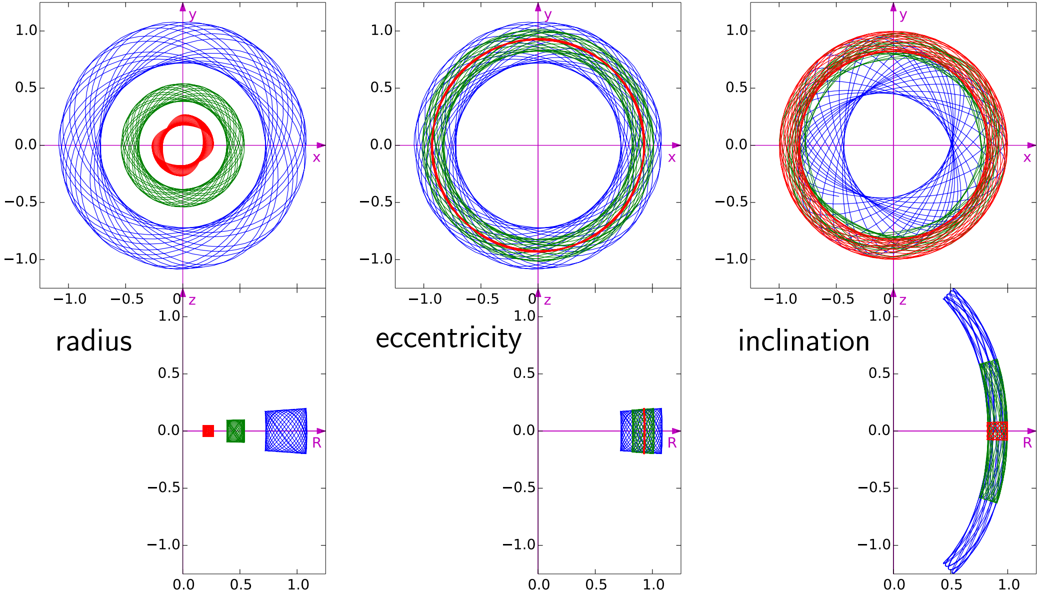


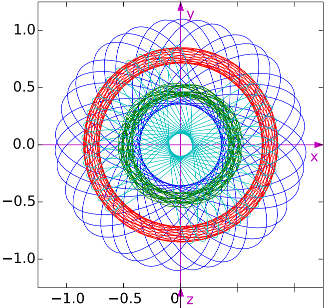
Orbits and integrals of motion

How large is the variety of orbits?

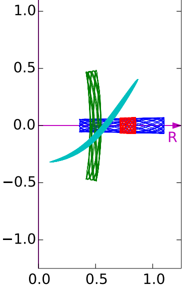
In general, *regular* orbits in *typical* stationary galactic potentials conserve 3 integrals of motion: one is always the energy E, and at a fixed energy, there are two more degrees of freedom (roughly corresponding to eccentricity and inclination).

The spherical case is degenerate in that it supports 4 integrals of motion (E and three components of the angular momentum $\mathbf{L}$).

In the axisymmetric case, the second integral is L_z, but the "non-classical" third integral I_3 (if exists), does not have an explicit expression (except in a special class of fully integrable potentials known as Stäckel potentials).



fixed E



Triaxial potentials may support up to two "non-classical integrals".

Not all orbits have the same number of integrals, and the physical meaning of these integrals is different between orbit families.

Summary

- Orbits of stars in galaxies are usually regular and belong to one of major families: boxes, loops (short- and long-axis).
- Nearly-circular orbits in axisymmetric discs can be treated within the epicyclic approximation.
- Poincaré surface of section (for 2d potentials) or frequency map (for 3d) are useful tools for analyzing ensembles of orbits supported by the potential.
- Once potential is specified, numerical orbit integration is trivial.