Modern stellar dynamics, lecture 5: action–angle variables

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Hamiltonian mechanics

Hamilton's equations of motion:

$$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \frac{\partial H(\mathbf{q},\mathbf{p})}{\partial \mathbf{p}},$$
$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\frac{\partial H(\mathbf{q},\mathbf{p})}{\partial \mathbf{q}},$$

where $\mathbf{q}$ are generalized coordinates, $\mathbf{p}$ are generalized momenta.

In the simplest case, $\mathbf{q}$ would be Cartesian coordinates $\mathbf{x} \equiv \{x, y, z\}$, and $\mathbf{p}$ – corresponding velocity components $\mathbf{v} \equiv \{v_x, v_y, v_z\}$.

$$H(\mathbf{x},\mathbf{v}) = \Phi(\mathbf{x}) + \tfrac{1}{2}|\mathbf{v}|^2.$$

Then we have to solve a system of coupled ODEs

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = p_i, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial \Phi(\mathbf{x})}{\partial x_i}.$$

Not particularly simple!

Hamiltonian mechanics

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where $\mathbf{q}$ are generalized coordinates, $\mathbf{p}$ are generalized momenta.

In the simplest case, $H(\mathbf{q},\mathbf{p}) = H(\mathbf{p})$, then

$$p_i(t) = \text{const}, \quad q_i(t) = \frac{\partial H}{\partial p_i}\, t + \text{const}.$$

This is *much* simpler, we have a complete solution at any moment of time!

But how to get there?

We need to find a suitable transformation from the usual position/velocity to the new variables.

Hamiltonian mechanics: Poisson brackets

Define the commutator operator for two functions of phase-space coordinates $A(\mathbf{q},\mathbf{p})$ and $B(\mathbf{q},\mathbf{p})$ as

$$[A,B] \equiv \frac{\partial A}{\partial \mathbf{q}}\frac{\partial B}{\partial \mathbf{p}} - \frac{\partial A}{\partial \mathbf{p}}\frac{\partial B}{\partial \mathbf{q}}.$$

It follows immediately that

$$[A,A] = 0, \quad [A,B] = -[B,A], \qquad \text{(antisymmetry)}$$

$$\big[[A,B],C\big] + \big[[B,C],A\big] + \big[[C,A],B\big] = 0, \qquad \text{(Jacobi identity)}$$

$$[q_i,q_j] = 0, \quad [p_i,p_j] = 0, \quad [q_i,p_j] = \delta_{ij}, \qquad i,j = 1..D,$$

and the Hamilton equations can be written as

$$\dot{q}_i = [q_i,H], \quad \dot{p}_i = [p_i,H]$$

Hamiltonian mechanics: integrals of motion

If $[A, B] = 0$, we say that A commutes with B.

If a function $A(\mathbf{q}, \mathbf{p})$ commutes with the Hamiltonian, it is conserved along the particle's trajectory – we call it an integral of motion:

$$\frac{dA}{dt} = \frac{\partial A}{\partial \mathbf{q}}\frac{d\mathbf{q}}{dt} + \frac{\partial A}{\partial \mathbf{p}}\frac{d\mathbf{p}}{dt}$$

$$= \frac{\partial A}{\partial \mathbf{q}}\frac{\partial H}{\partial \mathbf{p}} - \frac{\partial A}{\partial \mathbf{p}}\frac{\partial H}{\partial \mathbf{q}}$$

$$= [A, H] = 0$$

Obviously, the Hamiltonian itself is an integral of motion.

Two integrals A, B may or may not commute. For instance, in a 3d spherical system, the components of angular momentum commute with its magnitude, but not between themselves: $[L_x, L^2] = 0$, but $[L_x, L_y] = L_z$.

Hamiltonian mechanics: canonical transformations

Consider a change of variables from $\mathbf{p}, \mathbf{q}$ to $\mathbf{P}, \mathbf{Q}$, and express the Hamiltonian $H(\mathbf{P}, \mathbf{Q})$ or any other function in phase space in terms of the new variables.

If the new variables satisfy the canonical commutatation relations
$[Q_i, Q_j] = 0, \quad [P_i, P_j] = 0, \quad [Q_i, P_j] = \delta_{ij},$
such transformation is called canonical (or symplectic).

It also preserves

- Hamilton's equations of motion:
  $\dot{\mathbf{Q}}_i = [Q_i, H], \ \dot{\mathbf{P}}_i = [P_i, H];$
- more generally, all Poisson brackets:
  $\big[A(\mathbf{p}, \mathbf{q}), B(\mathbf{p}, \mathbf{q})\big] = \big[A(\mathbf{P}, \mathbf{Q}), B(\mathbf{P}, \mathbf{Q})\big];$
- all Poincaré invariants: $\oint \mathbf{p} \cdot d\mathbf{q};$
- $2D$-dimensional phase volume element: $d^D\mathbf{q}\, d^D\mathbf{p} = d^D\mathbf{Q}\, d^D\mathbf{P}.$

Examples of canonical transformations

1. Exchange: $\mathbf{Q} = \mathbf{p},\ \mathbf{P} = \mathbf{q}$
   (i.e., there is no fundamental difference between coordinate and momentum variables).
2. Point transformation: define $\mathbf{Q}(\mathbf{q})$ in whatever way, and then $\mathbf{P}(\mathbf{q}, \mathbf{p})$ is uniquely specified.

   For instance, cartesian to polar coordinates: $\mathbf{q} \equiv \{x,\ y\}$ to $\mathbf{Q} \equiv \{r,\ \phi\}$ implies $\mathbf{P} \equiv \{p_r,\ p_\phi\} = \big\{(xp_x + yp_y)/r,\ xp_y - yp_x\big\}$.
3. Hamiltonian flow: integrate the equations of motion for some time τ, and let $\{\mathbf{Q}, \mathbf{P}\}(\mathbf{q}, \mathbf{p};\ \tau)$ be the new coordinates and momenta of a point started from initial conditions $\mathbf{q}, \mathbf{p}$.
4. Action–angle variables $\mathbf{Q} \equiv \boldsymbol{\theta},\ \mathbf{P} \equiv \mathbf{J}$ defined for a separable Hamiltonian system that performs oscillatory motion in each dimension:
   $J_i = \dfrac{1}{2\pi} \oint p_i \, \mathrm{d}q_i,$ where the integration is performed over a closed contour in the i-th coordinate while keeping all other p_j, q_j fixed.

Action–angle variables for a 1d simple harmonic oscillator

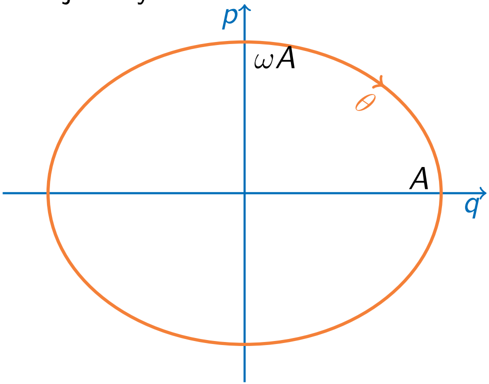
Hamiltonian: $H(q,p) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$.

The trajectory is $q(t) = A\sin(\omega t + \phi_0)$, $p(t) = A\omega\cos(\omega t + \phi_0)$, and the energy is $E = \frac{1}{2}\omega^2 A^2$.

The motion is periodic with frequency ω ($\Leftrightarrow$ period $2\pi/\omega$), so we define the angle $\theta = \omega t + \phi_0$.

The action J is $\frac{1}{2\pi}\times$area enclosed by the trajectory:

$$
\begin{aligned}
J &= \frac{1}{2\pi}\oint p\,dq \\
&= \frac{1}{2\pi}\int_0^{2\pi} p(\theta)\,\frac{dq}{d\theta}\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} A^2\omega\cos^2\theta\,d\theta \\
&= \frac{A^2\omega}{2} = \frac{E}{\omega}
\end{aligned}
$$

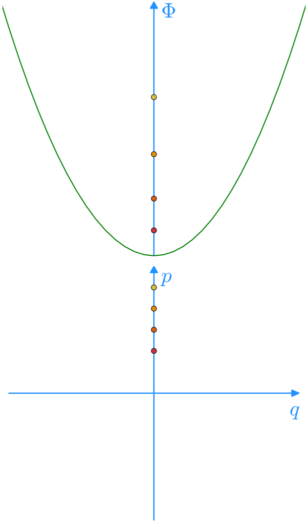


Action–angle variables for a generic 1d potential

For a generic 1d Hamiltonian $H(p, q) = \frac{1}{2}p^2 + \Phi(q)$, the action is still defined as

$$J = \frac{1}{2\pi} \oint p \, dq = \frac{2}{\pi} \int_0^{\Phi^{-1}(E)} \sqrt{2\left[E - \Phi(q)\right]} \, dq,$$

and the frequency $\Omega \equiv \dfrac{dH}{dJ}$ usually varies with energy.



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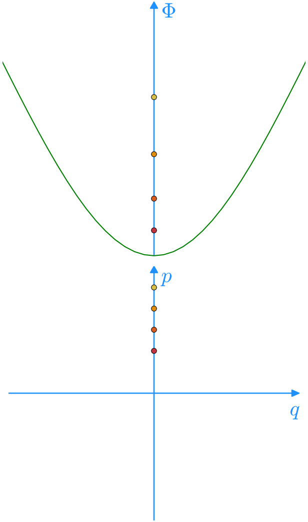

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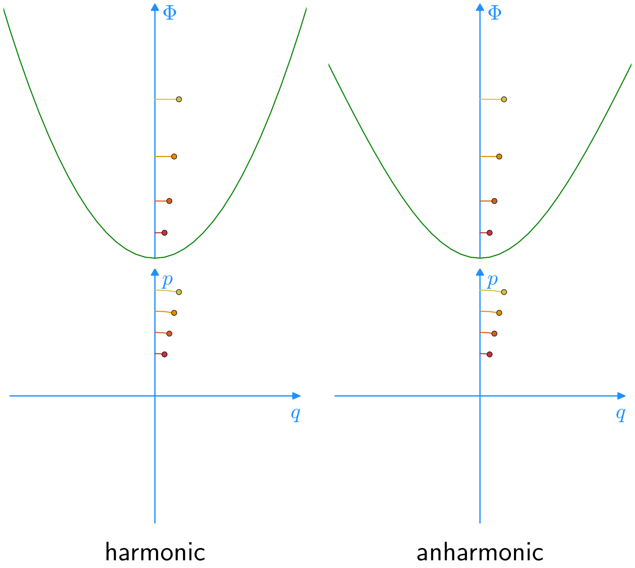

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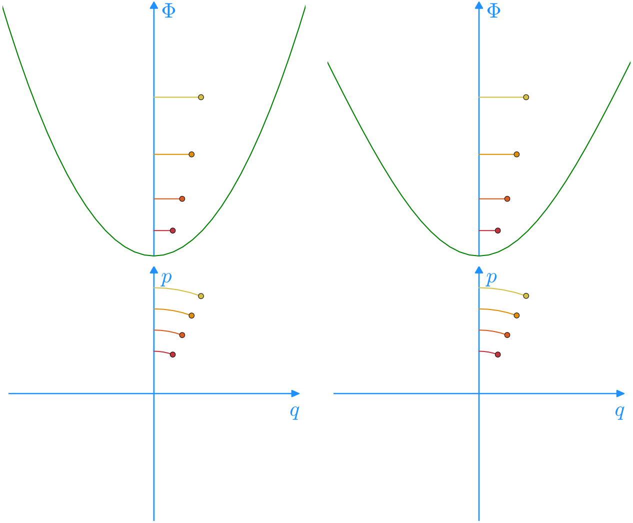

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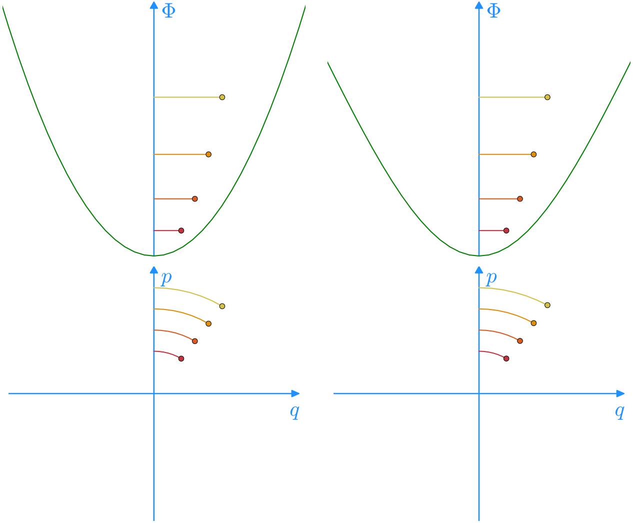

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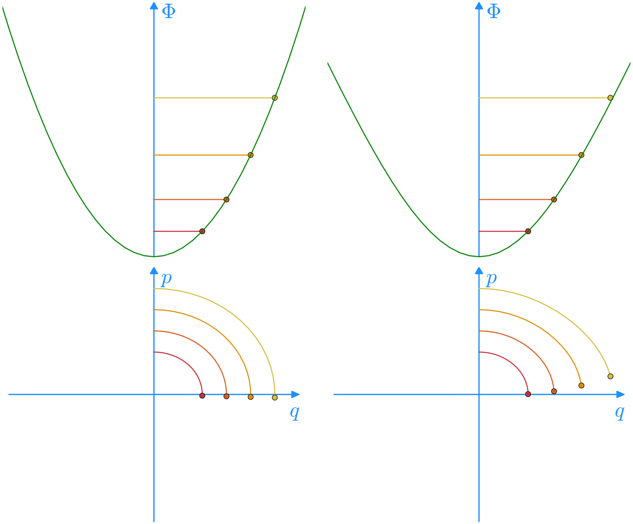

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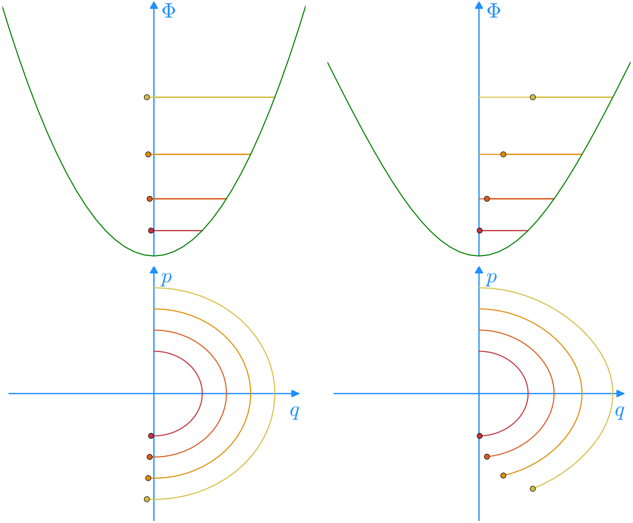

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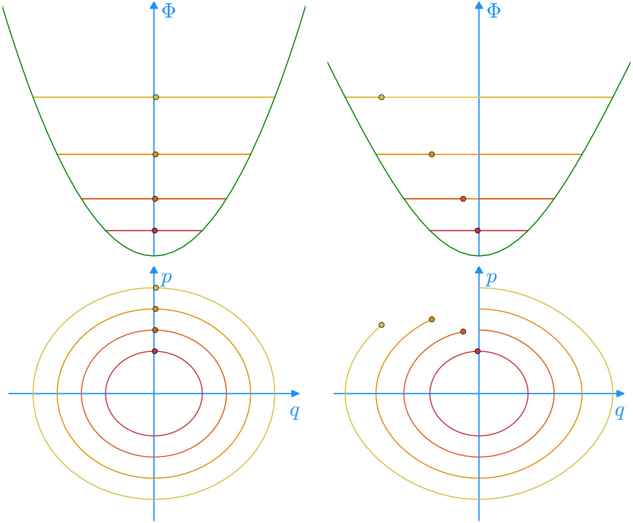

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Action–angle variables for a 2d simple harmonic oscillator

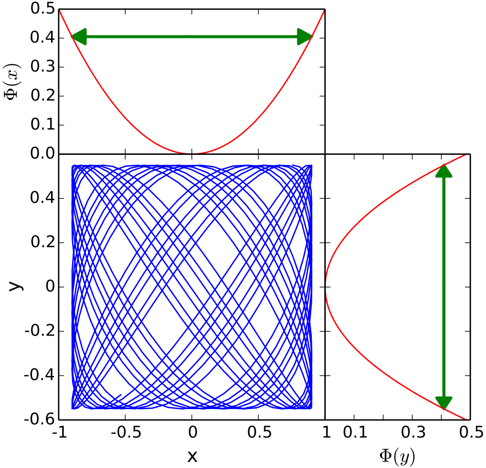
The same thing but in two dimensions: $\mathbf{q} = \{x, y\}, \ \ \mathbf{p} = \{p_x, p_y\};$

Hamiltonian:

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\left(p_x^2 + \omega_x^2 x^2\right) + \frac{1}{2}\left(p_y^2 + \omega_y^2 y^2\right)$$

$$\equiv \quad H_x(x, p_x) \quad + \quad H_y(y, p_y)$$

Motion is separable in x, y –
two uncoupled simple harmonic oscillators,
two integrals of motion E_x, E_y,
actions are $J_x = E_x/\omega_x, J_y = E_y/\omega_y$.



Action–angle variables for a 2d planar axisymmetric potential

A slightly more complicated system: two degrees of freedom, motion in an axisymmetric potential $\Phi(x, y) = \Phi(R)$, where $R \equiv \sqrt{x^2 + y^2}$.

Canonical coordinates: $\mathbf{q} = \{R, \phi\}, \mathbf{p} = \{p_R, p_\phi\}$

$$\text{Hamiltonian:}\quad H = \Phi(R) + \frac{1}{2}\left(p_R^2 + \frac{p_\phi^2}{R^2}\right) \equiv \Phi_{\text{eff}}(R) + \frac{1}{2}p_R^2$$

$$\text{equations of motion:}\quad \dot{R} = p_R, \quad \dot{\phi} = \frac{p_\phi}{R^2}, \quad \dot{p}_R = -\frac{d\Phi_{\text{eff}}}{dR}, \quad \dot{p}_\phi = 0$$

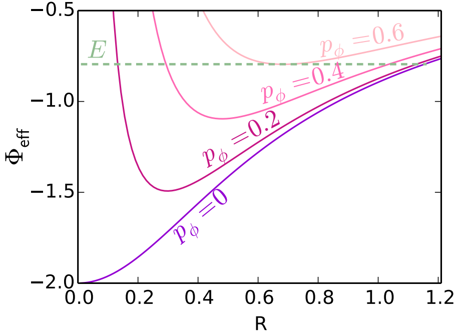
integrals of motion: E and p_ϕ

Motion in R is described by a 1d effective potential $\Phi_{\text{eff}}(R) \equiv \Phi(R) + p_\phi^2/R^2$

The radial action is

$$J_R = \frac{1}{\pi}\int_{R_-}^{R_+} p_R(R;\, E, p_\phi)\, dR$$

$$= \frac{1}{\pi}\int_{R_-}^{R_+} \sqrt{2\left[(E - \Phi_{\text{eff}}(R)\right]}\, dR$$



Action–angle variables for a 2d planar axisymmetric potential

Motion in ϕ: $\dot{p}_\phi = 0 \quad \Rightarrow \quad p_\phi = \text{const}$,

hence the azimuthal action is

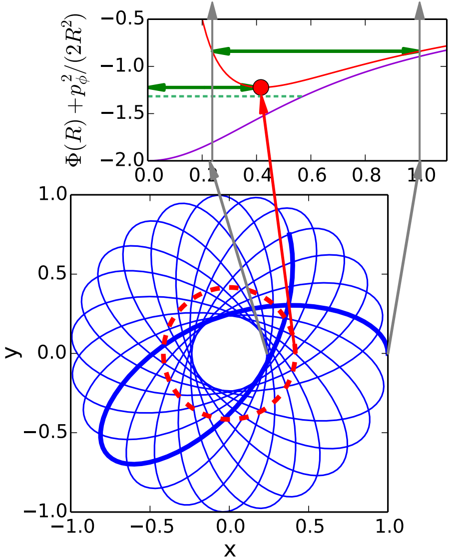
$$J_\phi = \frac{1}{2\pi} \int_0^{2\pi} p_\phi \, d\phi = p_\phi.$$

The actions J_R, J_ϕ describe the extent of the orbit in two complementary dimensions:

J_ϕ corresponds to the "guiding radius" (the radius of a circular orbit with the given angular momentum J_ϕ),

J_R gives the extent of radial oscillation about this guiding radius.

They can be varied independently, and any possible choice (provided that $J_R \geq 0$) corresponds to some trajectory.



Angles and frequencies

Note that $\dot{\phi} = p_\phi / R^2(t) \neq \text{const}$, so ϕ is not a canonically conjugate angle variable to p_ϕ!

Such variable is the azimuthal phase angle θ_ϕ, which increases linearly with time, and so does the radial phase angle θ_R:

$\theta_R = \Omega_R\, t, \quad \theta_\phi = \Omega_\phi\, t, \quad$ where

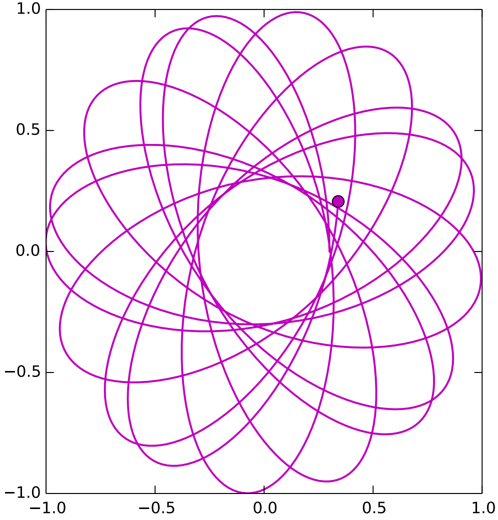
$$\Omega_R \equiv \frac{\partial H(J_R, J_\phi)}{\partial J_R}, \quad \Omega_\phi \equiv \frac{\partial H(J_R, J_\phi)}{\partial J_\phi} \quad \text{are orbital frequencies.}$$

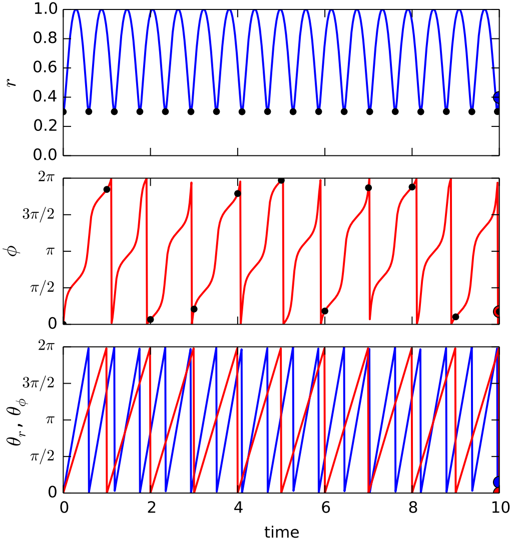
$$\text{Radial orbital period } T_R \equiv \frac{2\pi}{\Omega_R} = 2\int_{R_-}^{R_+} \frac{dR}{p_R} = 2\int_{R_-}^{R_+} \frac{dR}{\sqrt{2\left[E - \Phi(R)\right] - \frac{p_\phi^2}{R^2}}}$$

$$\text{Azimuthal period } T_\phi \equiv \frac{2\pi}{\Omega_\phi} = \frac{2\pi\ \int_{R_-}^{R_+} dR/p_R}{p_\phi\ \int_{R_-}^{R_+} dR/(R^2\, p_R)}$$

Example of actions in a 2d axisymmetric potential

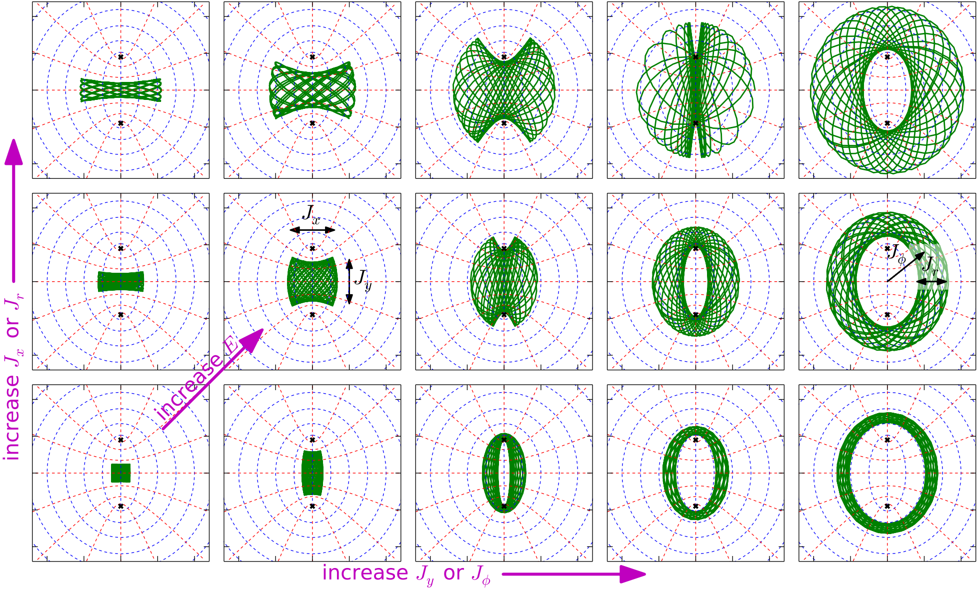
Radius R is uniquely related to the radial phase angle θ_R, which is typically set to zero at pericentre; however, the azimuthal angle ϕ increases non-uniformly with time (unlike θ_ϕ) and changes over one azimuthal period only *on average* by 2π.





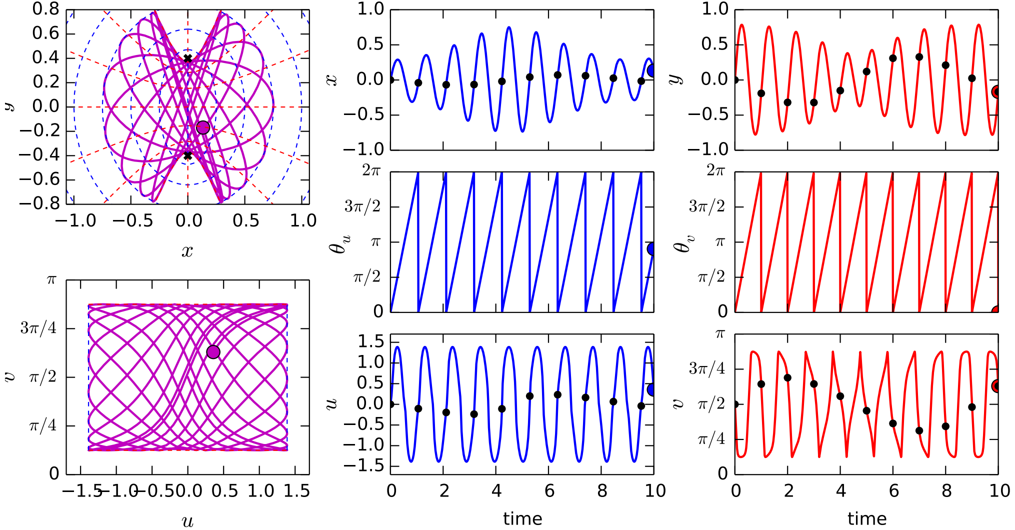
Actions in a general 2d non-axisymmetric potential

Transformation to action–angle variables is possible only when the motion is regular (e.g., in a Stäckel potential), but the physical meaning of actions depends on the orbit type.



Actions for a box orbit in a 2d non-axisymmetric potential

In the most general Stäckel case, the motion is separable in the elliptic coordinates u, v (the orbit fills a rectangle), but neither coordinate is uniquely linked to its phase angle $\theta_{u,v}$, nor is strictly periodic in time.



Action–angle variables for a 3d spherical potential

Spherical coordinates: $r, \theta, \phi, p_r, p_\theta, p_\phi$

Hamiltonian: $H = \Phi(r) + \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right)$

Integrals of motion: $E, L_x, L_y, L_z \left[, L \equiv \sqrt{L_x^2 + L_y^2 + L_z^2}\right]$

Radial action: $J_r = \frac{1}{\pi}\int_{r_-}^{r_+} \sqrt{2\left[E - \Phi(r)\right] - \frac{L^2}{r^2}}\, dr \quad \geq 0$

Azimuthal action: $J_\phi = L_z$ (any sign)

Vertical action: $J_\theta \equiv J_z = L - |L_z| \quad \geq 0$

In general, actions, angles, frequencies, or $H(\mathbf{J})$ do not have analytic expressions. One exception is the isochrone potential [Hénon 1959]:

$$\Phi(r) = -\frac{G\,M}{b + \sqrt{b^2 + r^2}}$$

(includes Kepler and harmonic oscliiator as limiting cases)

$$H(\mathbf{J}) = -\frac{2\,(G\,M)^2}{\left(2J_r + L + \sqrt{L^2 + 4\,G\,M\,b}\right)^2}$$

Action–angle variables for a 3d axisymmetric potential

In general, the transformation to action–angle variables is possible only for regular orbits and depends on the orbit type. For the practically important case of nearly-circular orbits close to the equatorial plane, the **epicyclic approximation** can be used: split the potential as $\Phi(R,z) \approx \Phi_R(R) + \Phi_z(z)$, and consider motion in R, ϕ as in the planar axisymmetric problem with an effective potential $\Phi_{\rm eff} = \Phi_R(R) + \frac{1}{2} L^2/r^2$, and independent, nearly harmonic motion in z.

Epicyclic frequencies:

azimuthal: $\Omega = \dfrac{v_{\rm circ}(R)}{R} = \sqrt{\dfrac{1}{R}\dfrac{\partial \Phi}{\partial R}}$,

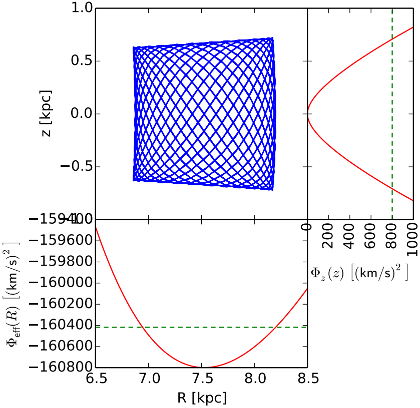
radial: $\kappa = \sqrt{\dfrac{\partial^2 \Phi_{\rm eff}}{\partial R^2}} = \sqrt{\dfrac{\partial^2 \Phi}{\partial R^2} + \dfrac{3}{R}\dfrac{\partial \Phi}{\partial R}}$,

vertical: $\nu = \sqrt{\dfrac{\partial^2 \Phi}{\partial z^2}}$,

and corresponding actions

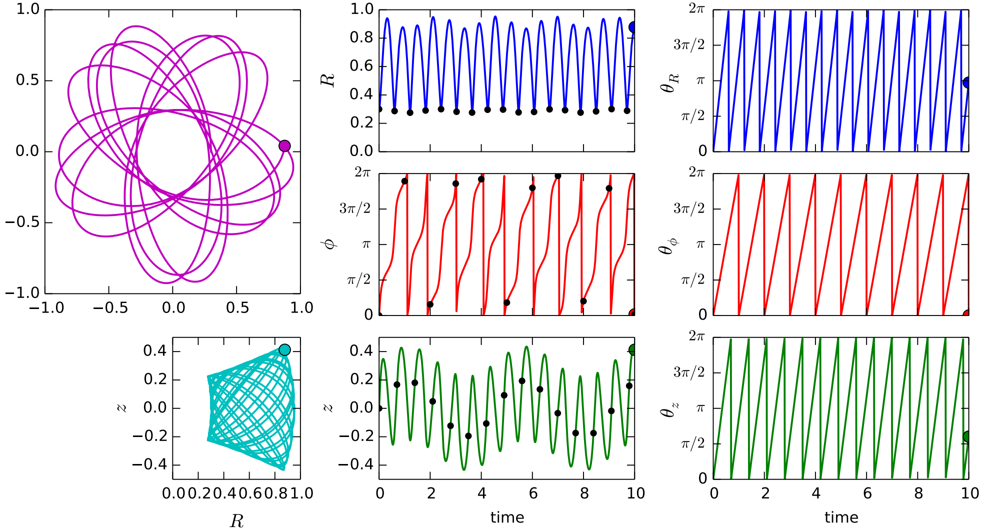
$J_\phi = L_z \equiv \Omega\, R^2,\;\; J_R = E_R/\kappa,\;\; J_z = E_z/\nu.$

However, it becomes increasingly inaccurate for orbits with high eccentricity and/or inclination.



Action–angle variables for a 3d axisymmetric potential

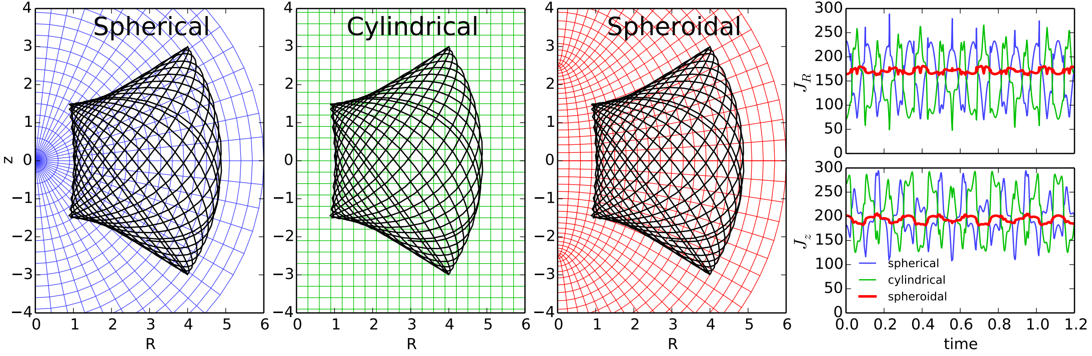
In a 3d flattened axisymmetric Stäckel potential, motion is separable in prolate ellipsoidal coordinates u, v, ϕ. The azimuthal action is still $J_\phi = L_z$, while radial and vertical actions J_R, J_z are given by 1d quadratures in u, v respectively. Note that the motion is not simply periodic in any single coordinate!



Action–angle variables for a 3d axisymmetric potential

Although realistic galactic potentials are not of the Stäckel form, most orbits are nevertheless rather well aligned with prolate spheroidal coordinates. One may exploit the assumption that the motion is separable in these coordinates (with an optimally chosen focal distance for each orbit) and compute the actions approximately – this is the "Stäckel Fudge" approach [Binney 2012]. These quantities are not strictly conserved along the orbit (hence are not true actions), but the error is typically $\lesssim 1-10\%$.

This is a reasonably accurate and efficient method for computing $\mathbf{J}, \boldsymbol{\theta}$ from $\mathbf{x}, \mathbf{v}$ (but not the other way round).



Action–angle computation with canonical transformations

Although the Stäckel approximation is nearly always sufficient in practice, its accuracy is "uncontrollable" (cannot be systematically improved) and it only works in one direction.

The full power of Hamiltonian mechanics in the *nonperturbative* regime is unleashed by canonical transformations in the action–angle space, enabling the computation of these quantities with an arbitrarily high accuracy as long as the motion remains regular.

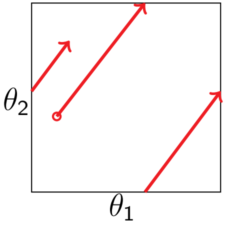
In brief, one introduces a "toy potential" Φ^{t}, in which the bi-directional transformation $\mathbf{x}, \mathbf{v} \Leftrightarrow \mathbf{J}^{\mathrm{t}}, \boldsymbol{\theta}^{\mathrm{t}}$ is known analytically (e.g., harmonic or isochrone potentials).
Then one constructs a canonical transformation between the true (yet unknown) $\mathbf{J}, \boldsymbol{\theta}$ and $\mathbf{J}^{\mathrm{t}}, \boldsymbol{\theta}^{\mathrm{t}}$, represented by a Fourier series in the periodic variables θ^{t} truncated at some adjustable order. The Fourier coefficients are obtained by minimizing the variation of the Hamiltonian over the extent of the approximated orbit.
The mapping $\mathbf{x}, \mathbf{v} \Rightarrow \mathbf{J}, \boldsymbol{\theta}$ relies on numerically integrating the orbit and then constructing a canonical transformation [Sanders&Binney 2014; Bovy 2014], while the reverse transformation ("torus mapping", McGill&Binney 1990; Kaasalainen 1994; Binney&McMillan 2016) provides the position/velocity at any time without integrating the orbit.
Both are considerably more expensive than the Stäckel fudge.

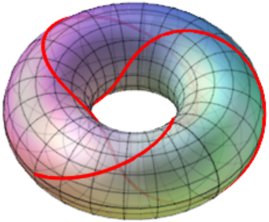
See Sanders&Binney 2016 for a general review.

Invariant tori

In an integrable potential, the motion is multiperiodic in angles: $\theta_i + 2\pi \cong \theta_i$, restricted to a D-dimensional hypersurface of the $2D$-dimensional phase space.

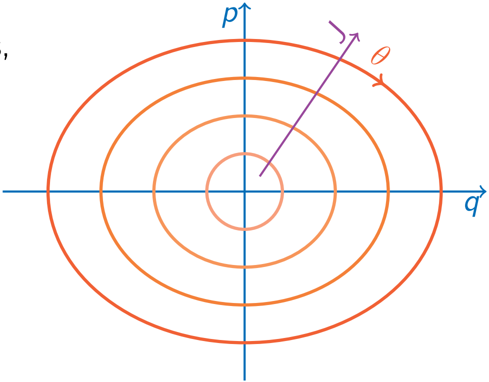
Arnold–Liouville theorem:
this hypersurface is diffeomorphic to (i.e., could be smoothly deformed into) a D-torus, parametrized by D periodic variables $\boldsymbol{\theta} \in [0..2\pi)$.





The entire 6d phase space is foliated into non-intersecting 3d orbital tori.

Actions tell you which orbit the star is on, angles – where it is located on this orbit.



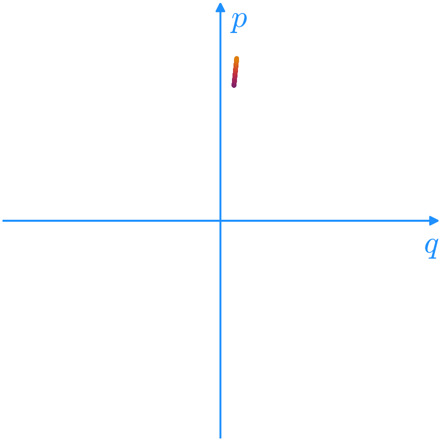
Invariant tori and phase mixing

Since the frequencies $\mathbf{\Omega} \equiv \dfrac{\partial H}{\partial \mathbf{J}} \neq \text{const}$, an initially localized ensemble of points eventually spreads out in angles (mixes in orbital phase) and fills the entire torus.

Thus in the time-averaged sense, only actions are important, and the distribution in angles is assumed to be uniform.

For chaotic orbits, the mixing is even more efficient.

However, at early stages of evolution (e.g., of a recently disrupted star cluster), the angle distribution is not yet mixed.



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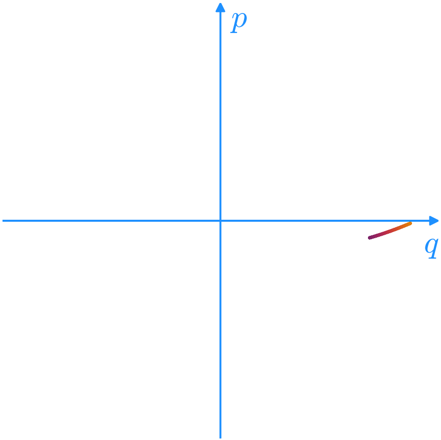

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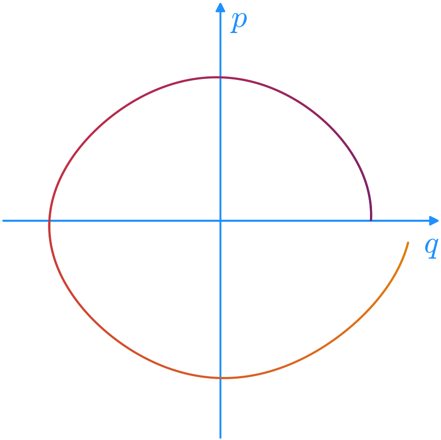

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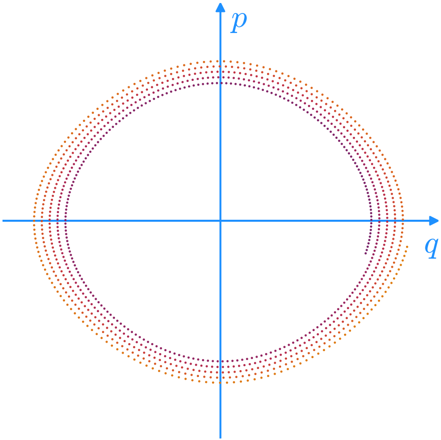
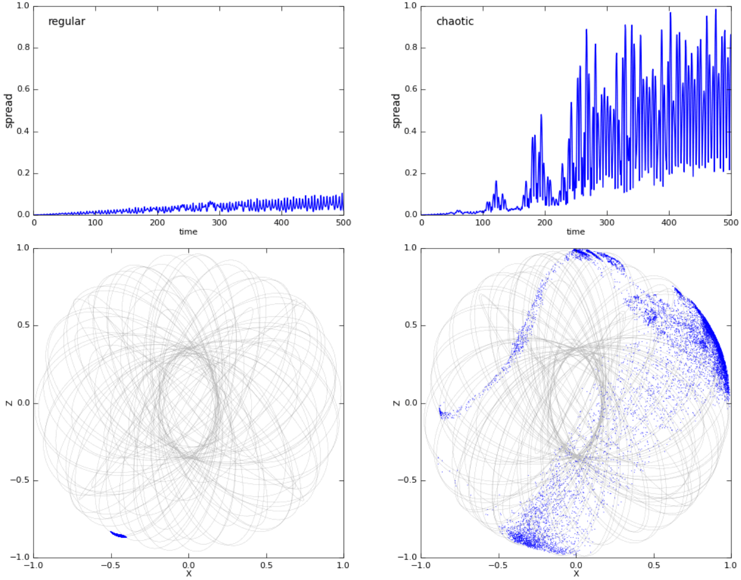


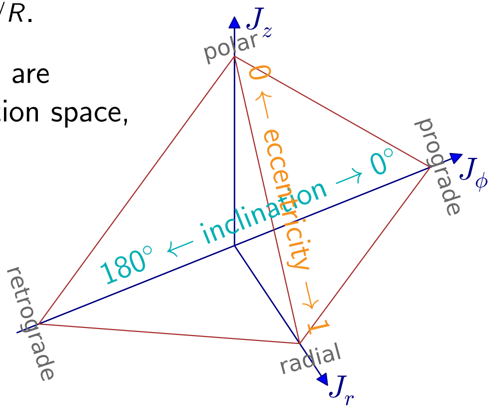
Phase mixing and chaotic mixing

For regular motion, the ensemble of nearby orbits drifts apart linearly with time; for chaotic, the spread grows as $\exp(\lambda t)$, where λ is the Lyapunov exponent.



Rules of thumb about actions

- Dimension of actions is length×velocity:
  if a star at a galactocentric distance r travels with velocity v, then [at least one of the actions] $J \sim r\,v$.
- Frequencies: $\Omega_i(\mathbf{J}) = \partial H / \partial J_i$
  characteristic velocity $v_i \sim \sqrt{\Omega_i\, J_i}$
  e.g., for a circular orbit $J_\phi = R\,v_\phi$, $\Omega_\phi = v_\phi / R$.
- Surfaces of constant energy $H(\mathbf{J}) = E$ are approximately tetrahedra in the 3d action space, with $E \approx E(\Omega_r\, J_r + \Omega_z\, J_z + \Omega_\phi\, J_\phi)$.



Summary: advantages of action–angle variables

- Actions have a clear physical meaning (describe the extent of oscillations in each dimension).
- Angles most naturally describe the motion (change linearly with time).
- Possible range for each action variable is $[0..\infty)$ or $(-\infty..\infty)$, independently of the other ones (unlike E and L, say).
- Canonical coordinates $\Rightarrow$ the 6d phase-space volume element is $d^3x\, d^3v = d^3J\, d^3\theta$.
- Actions are adiabatic invariants (are conserved under slow variation of potential).
- Perturbation theory most naturally formulated in terms of actions.
- Efficient methods for conversion between $\{\mathbf{x}, \mathbf{v}\}$ and $\{\mathbf{J}, \boldsymbol{\theta}\}$ now exist; in oblate axisymmetric potentials, the Stäckel approximation works well.