Modern stellar dynamics, lecture 6:

collisionless Boltzmann equation and its moments

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Part III / MAst course, Winter 2022

Distribution function in stellar dynamics

A stellar system composed of a large number $N \gg 1$ of "identical" stars can be described by a distribution function (DF):

$f(\mathbf{x}, \mathbf{v};\; t)$ – probability density in the phase space (at time t),

$\int f(\mathbf{x}, \mathbf{v})\, \mathrm{d}^3x\, \mathrm{d}^3v = 1$ (unit-normalized), or alternatively

$\int f(\mathbf{x}, \mathbf{v})\, \mathrm{d}^3x\, \mathrm{d}^3v = M_{\mathrm{total}}$ (mass-normalized).

For a multicomponent system, define a separate DF f_k for each species k, or an "extended distribution function" (EDF) with additional arguments:

$f(\mathbf{x}, \mathbf{v};\; \boldsymbol{\eta};\; t)$ ($\boldsymbol{\eta}$ may be age, mass, chemical composition, etc.)

Properties of the distribution function

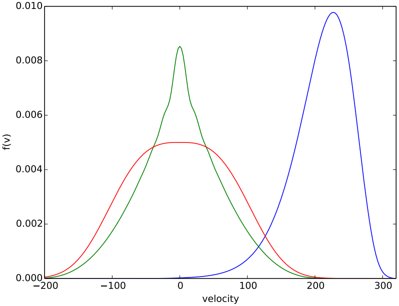
DF offers a complete description of the stellar population:

- density $\rho(\mathbf{x}) = \int f(\mathbf{x},\mathbf{v})\,\mathrm{d}^3v$,
- mean velocity $\overline{\mathbf{v}}(\mathbf{x}) = \dfrac{1}{\rho(\mathbf{x})}\int \mathbf{v}\, f(\mathbf{x},\mathbf{v})\,\mathrm{d}^3v$,
- second moment of velocity $\overline{v^2_{ij}}(\mathbf{x}) = \dfrac{1}{\rho(\mathbf{x})}\int v_i\, v_j\, f(\mathbf{x},\mathbf{v})\,\mathrm{d}^3v$,

  velocity dispersion tensor $\sigma^2_{ij} \equiv \overline{v^2_{ij}} - \overline{v_i}\,\overline{v_j}$,
- more generally, velocity distribution at a given point

  $\mathfrak{f}(v_1;\,\mathbf{x}) = \dfrac{1}{\rho(\mathbf{x})}\int f(\mathbf{x},\mathbf{v})\,\mathrm{d}v_2\,\mathrm{d}v_3$

  (note that it may be rather non-Gaussian!),
- etc.

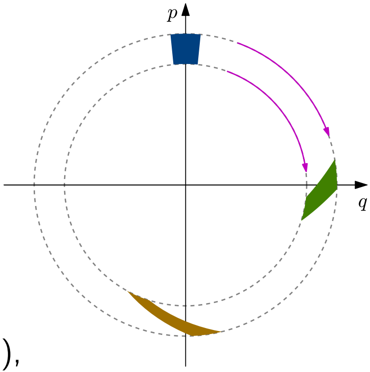


Evolution of the distribution function

As the system evolves, stars move along their trajectories, but the DF at the location of any star is conserved (Liouville's theorem):

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \left[f, H\right] = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}}\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} + \frac{\partial f}{\partial \mathbf{v}}\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}$$

$$= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{x}}\,\mathbf{v} - \frac{\partial f}{\partial \mathbf{v}}\frac{\partial \Phi}{\partial \mathbf{x}} = 0$$

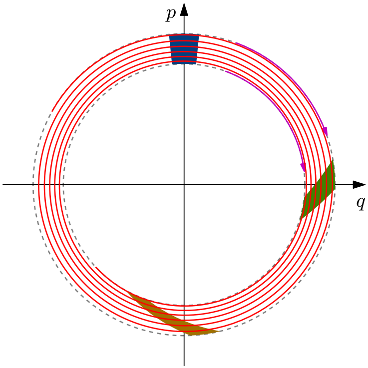


Collisionless Boltzmann (or Vlasov) equation (CBE), generally valid for large-N systems (e.g., entire galaxies), in which the relaxation time is long compared to their age (even if the system is not in a steady state!).

Note that CBE looks the same in any canonical coordinates $\mathbf{q}, \mathbf{p}$.

Phase mixing and the Jeans theorem

In realistic galactic potentials, orbital frequencies $\mathbf{\Omega} \equiv \dfrac{\partial H}{\partial \mathbf{J}} \neq$ const, hence an initially localized ensemble of points eventually attains a uniform distribution in phase angles $\boldsymbol{\theta}$ and fills the entire accessible region for its integrals of motion.



Jeans theorem

In a steady state, the DF may depend only on the integrals of motion:

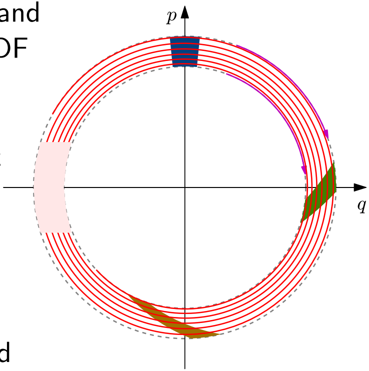
$$f(\mathbf{x},\mathbf{v}) \;=\; f\big(\mathcal{I}(\mathbf{x},\mathbf{v};\,\Phi)\big)$$

Here $\mathcal{I}$ may be energy E, or actions $\mathbf{J}$ – anything that is conserved by the given potential.

As the phase mixing timescale is usually $\gg \Omega^{-1}$, the apparently well-mixed state of galaxies is likely caused by other processes, such as *violent relaxation* in a rapidly varying potential.

Limitations of CBE

- Deals with the DF $f(\mathbf{x}, \mathbf{v})$ in the continuum limit (ignores that the system is actually composed of discrete stars).
- Mixing conserves the DF but stretches it into thinner and thinner filaments – at some point, this "fine-graned" DF becomes meaningless.
  In this illustration, the phase space consists of interleaving stripes with $f = 0$ and $f = 1$, but if we cannot resolve these filaments, we will measure the "coarse-grained" DF somewhere between 0 and 1.
  However, there is no equation describing the evolution of the coarse-grained DF.



- CBE also implies that stars move in the mean field and do not feel the force from each other individually. Over long timescales, two-body encounters between stars cannot be ignored anymore, and we need to consider a more general equation with a collision term in the right-hand side (generalized Landau equation or its simplifications, including various forms of Fokker–Planck equation – relevant for thermodynamically old systems such as globular clusters).

Legacy of Sir James Jeans

- Jeans **theorem**: The DF is a function of 6 phase-space coordinates, but in a steady state it may only depend on the integrals of motion (i.e., is reduced to a function of at most 3 variables).
- Jeans **equations** ignore this simplification and instead look at the moments of the DF $f(\mathbf{x}, \mathbf{v}, t)$ w.r.t. velocity:

  $$\rho(\mathbf{x},\ t) \equiv \int f(\mathbf{x}, \mathbf{v},\ t)\ \mathrm{d}^3 v, \quad \rho\,\overline{v_i} \equiv \int f(\mathbf{x}, \mathbf{v},\ t)\ v_i\ \mathrm{d}^3 v, \quad \text{etc.}$$

  They were derived from the CBE by Maxwell and are analogous to the Euler equations.

  That's another massive simplification, reducing the full 6d DF into a small number of functions of 3d coordinates only (possibly time-dependent).
- Jeans length, mass and instability – will be considered elsewhere.
- Rayleigh–Jeans law of blackbody radiation – not part of this course.
- Formation scenario of the Solar system via a close encounter with a passing star – didn't stand up to scrutiny.

Jeans equations

Multiply the CBE by various powers of velocity and integrate over velocities.

A simple 1d example:

CBE for $f(x, v_x;\ t)$: $$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\, v_x - \frac{\partial f}{\partial v_x}\,\frac{\partial \Phi}{\partial x} = 0$$

Jeans equations

Multiply the CBE by various powers of velocity and integrate over velocities.

A simple 1d example:

$$\int_{-\infty}^{\infty}\left[\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x}\,v_x-\frac{\partial f}{\partial v_x}\,\frac{\partial \Phi}{\partial x}\right]\mathrm{d}v_x=0$$

integrate by parts

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty}f(x,v_x)\,\mathrm{d}v_x+\frac{\partial}{\partial x}\int_{-\infty}^{\infty}f(x,v_x)\,v_x\,\mathrm{d}v_x-\frac{\partial\Phi(x)}{\partial x}\Big(f(x,v_x)\Big)\Big|_{v_x=-\infty}^{+\infty}=0$$

Jeans equations

Multiply the CBE by various powers of velocity and integrate over velocities.

A simple 1d example:

$$\int_{-\infty}^{\infty} \left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} v_x - \frac{\partial f}{\partial v_x} \frac{\partial \Phi}{\partial x} \right] \mathrm{d} v_x = 0$$

integrate by parts

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} f(x, v_x) \, \mathrm{d} v_x + \frac{\partial}{\partial x} \int_{-\infty}^{\infty} f(x, v_x) \, v_x \, \mathrm{d} v_x - \frac{\partial \Phi(x)}{\partial x} \cancel{\Big(f(x, v_x) \Big)} \Big|_{v_x = -\infty}^{+\infty} = 0$$

vanishes because

$f \rightarrow 0$ as $|v_x| \rightarrow \infty$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \left(\rho \, \overline{v_x} \right)}{\partial x} = 0$$

continuity equation

Jeans equations

Multiply the CBE by various powers of velocity and integrate over velocities.

A simple 1d example:

multiply by v_x :

$$\int_{-\infty}^{\infty}\left[\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x}\,v_x-\frac{\partial f}{\partial v_x}\,\frac{\partial \Phi}{\partial x}\right]v_x\,\mathrm{d}v_x=0$$

integrate by parts

$$\frac{\partial(\rho\,\overline{v_x})}{\partial t}+\frac{\partial\left(\rho\,\overline{v_x^2}\right)}{\partial x}-\frac{\partial \Phi}{\partial x}\left[\cancel{\left(f\,v_x\right)\Big|_{v_x=-\infty}^{+\infty}}-\int_{-\infty}^{\infty} f\,\underbrace{\frac{\partial v_x}{\partial v_x}}_{=1}\mathrm{d}v_x\right]=0$$

vanishes because

$f\ v_x \rightarrow 0$ as $|v_x| \rightarrow \infty$

Jeans equations

Multiply the CBE by various powers of velocity and integrate over velocities.

A simple 1d example:

multiply by v_x :

$$\int_{-\infty}^{\infty}\left[\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x}\,v_x-\frac{\partial f}{\partial v_x}\,\frac{\partial \Phi}{\partial x}\right]v_x\,\mathrm{d}v_x=0$$

$$\frac{\partial(\rho\,\overline{v_x})}{\partial t}+\frac{\partial\left(\rho\,\overline{v_x^2}\right)}{\partial x}+\frac{\partial\Phi}{\partial x}\,\rho=0$$

then expand $\overline{v_x^2}=\sigma_x^2+\overline{v_x}^2$

subtract $\overline{v_x}\left[\frac{\partial\rho}{\partial t}+\frac{\partial\left(\rho\,\overline{v_x}\right)}{\partial x}\right]=0$ (the continuity equation)

$$\Downarrow$$

$$\rho\,\frac{\partial\overline{v_x}}{\partial t}+\rho\,\overline{v_x}\,\frac{\partial\overline{v_x}}{\partial x}+\frac{\partial(\rho\,\sigma_x^2)}{\partial x}+\frac{\partial\Phi}{\partial x}\,\rho=0$$

analog of Euler equation

gravitational force

pressure gradient

when $\overline{v_x}=0$, describes hydrostatic equilibrium

Jeans equations (general form in 3d Cartesian coords)

Multiply the CBE by *i*-th velocity component and integrate over 3d velocity:

$$
\begin{aligned}
0 &= \int \mathrm{d}^3\mathbf{v}\ v_i \left\{ \frac{\partial f(\mathbf{x},\mathbf{v})}{\partial t} + \sum_j \left[v_j \frac{\partial f(\mathbf{x},\mathbf{v})}{\partial x_j} - \frac{\partial \Phi(\mathbf{x})}{\partial x_j} \frac{\partial f(\mathbf{x},\mathbf{v})}{\partial v_j} \right] \right\} \\
&= \frac{\partial}{\partial t} \int \mathrm{d}^3\mathbf{v}\, f\, v_i + \sum_j \left[\int \mathrm{d}^3\mathbf{v}\ v_i\, v_j \frac{\partial f}{\partial x_j} - \frac{\partial \Phi}{\partial x_j} \int \mathrm{d}^3\mathbf{v}\ v_i \frac{\partial f}{\partial v_j} \right]
\end{aligned}
$$

integrate by parts

$$
\begin{aligned}
&= \frac{\partial(\rho\, \overline{v_i})}{\partial t} + \sum_j \left[\frac{\partial}{\partial x_j} \int \mathrm{d}^3\mathbf{v}\ v_i\, v_j\, f + \frac{\partial \Phi}{\partial x_j} \int \mathrm{d}^3\mathbf{v} \underbrace{\frac{\partial v_i}{\partial v_j}}_{=\delta_{ij}} f \right] \\
&= \frac{\partial(\rho\, \overline{v_i})}{\partial t} + \sum_j \frac{\partial\left(\rho\ \overline{v_i\, v_j}\right)}{\partial x_j} + \frac{\partial \Phi}{\partial x_i}\, \rho,
\end{aligned}
$$

subtract $\quad 0 = v_i \left\{ \frac{\partial \rho}{\partial t} + \sum_j \frac{\partial(\rho\, \overline{v_j})}{\partial x_j} \right\}$ (the continuity equation)

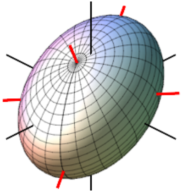
finally get $\quad 0 = \rho \frac{\partial \overline{v_i}}{\partial t} + \sum_j \left[\rho\, \overline{v_j} \frac{\partial \overline{v_i}}{\partial x_j} + \frac{\partial(\rho\, \sigma_{ij}^2)}{\partial x_j} \right] + \rho \frac{\partial \Phi}{\partial x_i}.$

Jeans equations (general form in 3d Cartesian coords)

$$\rho\,\frac{\partial \overline{v_i}}{\partial t} + \sum_j \left[\rho\,\overline{v_j}\,\frac{\partial \overline{v_i}}{\partial x_j} + \frac{\partial(\rho\,\sigma_{ij}^2)}{\partial x_j} \right] + \rho\,\frac{\partial \Phi}{\partial x_i} = 0, \quad i = 1..3.$$

gradient of pressure (stress) tensor

gravitational force



not a complete analogue of the Euler equation:
the latter contains a scalar pressure variable, while in collisionless systems σ_{ij} is a symmetric 3×3 tensor with 6 independent components ("velocity ellipsoid").

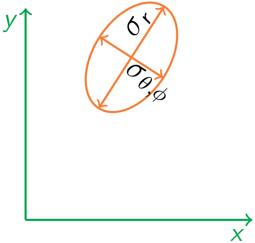
In fluid dynamics, density ρ, velocity vector $\mathbf{v}$ and pressure P (5 unknowns) are related by 5 equations (continuity, 3 components of momentum conservation, and equation of state).

In gravitational dynamics, we have 4 equations for 10 unknowns, so in general cannot uniquely determine even the low-order moments of CBE in the given Φ.

OTOH, *if we know* ρ, $\bar{\mathbf{v}}$ and σ_{ij}, we may determine the gravitational potential from the Jeans equations – this is the standard way of using them.
The problem is that we can rarely measure all components of the velocity ellipsoid, and thus have to make simplifying assumptions, which may strongly affect the result.

Jeans equations: spherical case

In the spherical non-rotating case, the velocity ellipsoid is aligned with the radial coordinate and has two independent components: radial σ_r^2 and tangential σ_t^2 dispersions, and only one nontrivial equation remains:



$$0 = \frac{\mathrm{d}\left(\rho\, \sigma_r^2\right)}{\mathrm{d}r} + \frac{\mathrm{d}\Phi}{\mathrm{d}r}\,\rho + \frac{\rho}{r}\left(2\sigma_r^2 - \underbrace{\left[\sigma_\theta^2 + \sigma_\phi^2\right]}_{\sigma_t^2}\right)$$

$$= \frac{\mathrm{d}\left(\rho\, \sigma_r^2\right)}{\mathrm{d}r} + \frac{\mathrm{d}\Phi}{\mathrm{d}r}\,\rho + \frac{2\beta}{r}\,\rho\, \sigma_r^2$$

where $\beta(r) \equiv 1 - \dfrac{\sigma_\theta^2(r) + \sigma_\phi^2(r)}{2\sigma_r^2(r)}$ is the anisotropy coefficient:

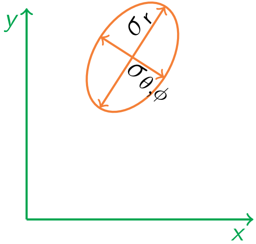
$\beta = 1$ – purely radial orbits,
$\beta > 0$ – radially anisotropic case,
$\beta = 0$ – isotropic case,
$\beta < 0$ – tangentially anisotropic case,
$\beta = -\infty$ – purely circular orbits.

Jeans equations: spherical case

In the spherical non-rotating case, the velocity ellipsoid is aligned with the radial coordinate and has two independent components: radial σ_r^2 and tangential σ_t^2 dispersions, and only one nontrivial equation remains:

$$0 = \frac{\mathrm{d}\left(\rho\;\sigma_r^2\right)}{\mathrm{d}r} + \frac{\mathrm{d}\Phi}{\mathrm{d}r}\,\rho + \frac{\rho}{r}\left(2\sigma_r^2 - \underbrace{\left[\sigma_\theta^2 + \sigma_\phi^2\right]}_{\sigma_t^2}\right)$$

$$= \frac{\mathrm{d}\left(\rho\;\sigma_r^2\right)}{\mathrm{d}r} + \frac{\mathrm{d}\Phi}{\mathrm{d}r}\,\rho + \frac{2\beta}{r}\,\rho\;\sigma_r^2$$



The solution for σ_r with the given $\beta(r)$ profile is given by

$$\sigma_r^2(r) = \frac{1}{\rho(r)\,g(r)}\int_r^\infty \mathrm{d}s\;\rho(s)\,g(s)\,\frac{\mathrm{d}\Phi(s)}{\mathrm{d}s}, \quad g(r) \equiv \exp\left[2\int_0^r \frac{\beta(s)}{s}\,\mathrm{d}s\right],$$

or for $\beta = \text{const}$, $\sigma_r^2(r) = \dfrac{1}{\rho(r)\,r^{2\beta}}\displaystyle\int_r^\infty \mathrm{d}s\;\rho(s)\,s^{2\beta}\,\frac{\mathrm{d}\Phi(s)}{\mathrm{d}s}$.

Jeans equations: axisymmetric case

$\Phi(R, z)$ – total gravitational potential

$\rho(R, z)$ – density of tracers

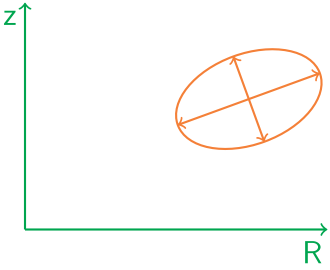
$$0 = \rho \frac{\partial \Phi}{\partial z} + \frac{\partial(\rho \, \sigma_z^2)}{\partial z} + \frac{\partial(\rho \, \overline{v_R v_z})}{\partial R} + \frac{\rho \, \overline{v_R v_z}}{R}$$

$$0 = \rho \frac{\partial \Phi}{\partial R} + \frac{\partial(\rho \, \sigma_R^2)}{\partial R} + \frac{\partial(\rho \, \overline{v_R v_z})}{\partial z} + \frac{\rho \left(\sigma_R^2 - \overline{v_\phi^2}\right)}{R}$$

Two equations for four unknown functions (components of the velocity ellipsoid tensor):

$\sigma_R^2, \;\; \sigma_z^2, \;\; \overline{v_R v_z}, \;\; \overline{v_\phi^2} = \overline{v_\phi}^2 + \sigma_\phi^2.$

Need further assumptions about the orientation of the velocity ellipsoid in the meridional plane.



Jeans equations: axisymmetric case – semi-isotropic

Assume $\overline{v_R v_z} = 0$ and $\sigma_R^2 = \sigma_z^2$:

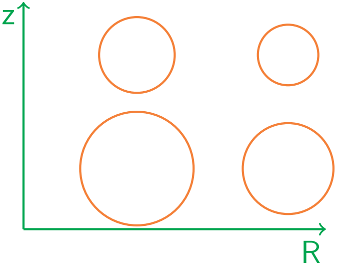
$$0 = \rho \frac{\partial \Phi}{\partial z} + \frac{\partial(\rho\, \sigma_R^2)}{\partial z}$$

$$0 = \rho \frac{\partial \Phi}{\partial R} + \frac{\partial(\rho\, \sigma_R^2)}{\partial R} + \frac{\rho\, \left(\sigma_R^2 - \overline{v_\phi^2}\right)}{R}$$

Used in many papers throughout 1980s – 2000s

Still need to decide* how to split $\overline{v_\phi^2} = \overline{v_\phi}^2 + \sigma_\phi^2$
e.g., assume full isotropy $\sigma_\phi^2 = \sigma_R^2$ (unrealistic!)

* this is true for all variants of Jeans equations



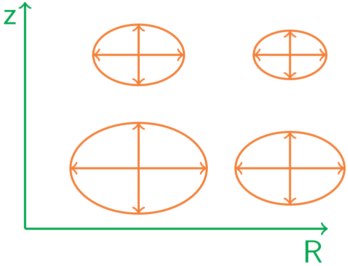
Jeans equations: axisymmetric case – cylindrical alignment

Assume $\overline{v_R v_z} = 0$ and $\sigma_R^2/\sigma_z^2 = b = \text{const}$:

$$0 = \rho \frac{\partial \Phi}{\partial z} + \frac{\partial(\rho\, \sigma_R^2)}{\partial z} \frac{1}{b}$$

$$0 = \rho \frac{\partial \Phi}{\partial R} + \frac{\partial(\rho\, \sigma_R^2)}{\partial R} + \frac{\rho\, \left(\sigma_R^2 - \overline{v_\phi^2}\right)}{R}$$

Jeans Anisotropic Method (JAM)
[Cappellari 2008; Watkins+ 2013]:
widely used in extragalactic studies due to its simplicity and low computational cost, though the assumption of a constant b is most likely violated in real galaxies.



Jeans equations: axisymmetric case – spherical alignment

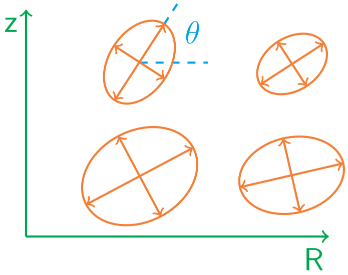
Assume orientation of the velocity ellipsoid towards the galactic center:

$$\tan 2\theta = \frac{2\,\overline{v_R v_z}}{\sigma_R^2 - \sigma_z^2} \;=\; \frac{2\,Rz}{R^2 - z^2}$$

$$0 = \rho\,\frac{\partial\Phi}{\partial z} + \frac{\partial(\rho\,\sigma_z^2)}{\partial z} + \frac{\partial(\rho\,\overline{v_R v_z})}{\partial R} + \frac{\rho\,\overline{v_R v_z}}{R}$$

$$0 = \rho\,\frac{\partial\Phi}{\partial R} + \frac{\partial(\rho\,\sigma_R^2)}{\partial R} + \frac{\partial(\rho\,\overline{v_R v_z})}{\partial z} + \frac{\rho\,\left(\sigma_R^2 - \overline{v_\phi^2}\right)}{R}$$

A good approximation for realistic galaxies;
advocated by Binney 2014; Evans+ 2016
but more complicated and rarely used
(although see Cappellari 2019);
need further assumptions about the shape
of the velocity ellipsoid.



Caveats of Jeans equations: no guarantee of a valid DF

$\gamma = 1$ Dehnen (aka Hernquist) model: $\Phi(r) = -\frac{G\,M}{r+a}, \quad \rho(r) = \frac{M\,a}{2\pi\,r\,(r+a)^3}.$

Consider the case of constant $\beta = 1/2$:

$$\sigma_r^2 = \frac{1}{\rho\,r^{2\beta}} \int_r^\infty \mathrm{d}s\; s^{2\beta}\,\rho(r)\,\frac{\mathrm{d}\Phi}{\mathrm{d}r} \;=\; \frac{r\,(r+a)^3}{r} \int_r^\infty \mathrm{d}r\;\frac{r}{r\,(r+a)^3}\,\frac{G\,M}{(r+a)^2} \;=\; \frac{G\,M}{4(r+a)}.$$

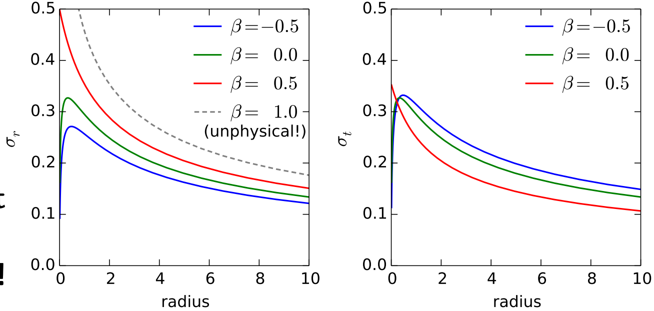
This matches the dispersion profile produced by a real DF

(discussed in the next lecture) $f(E,L) = \frac{3a\,E^2}{4\pi^3\,G^3\,M^3\,L}.$

Now try $\beta = 1$: Jeans equation gives $\sigma_r^2 = \frac{(4r+a)\,G\,M}{12\,r\,(r+a)}.$

This diverges as $r \to 0$, but $\Phi(0)$ is finite, so σ must also remain finite.

The Jeans solution does not necessarily correspond to a physically plausible model!



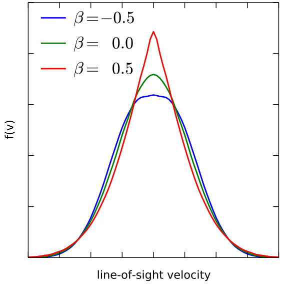
Caveats of Jeans equations: mass–anisotropy degeneracy

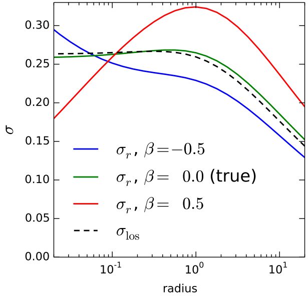
In external galaxies, we usually measure only the projected quantities – surface density $\Sigma(R)$ and line-of-sight velocity dispersion

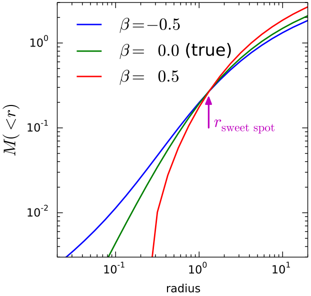
$$\sigma_{\rm los}^2(R) = \frac{2}{\Sigma(R)} \int_R^\infty \left(1 - \beta(r)\,\frac{R^2}{r^2}\right)\ \frac{\sigma_r^2(r)\,\rho(r)\,r}{\sqrt{r^2 - R^2}}\,dr \quad \text{[Binney \& Mamon 1982]}.$$

Different assumptions about $\beta(r)$ lead to different deprojected $\sigma_r(r)$ and hence to different inferred mass profiles, although the mass at some "sweet-spot radius" ($\simeq$ half-light radius of tracers) is insensitive to β [Wolf+ 2010].

This degeneracy is partially lifted for DF-based methods, which can utilize the full line-of-sight velocity distribution, not just its dispersion.







Virial equations

If we multiply Jeans equations by x_k and integrate over the 3d space, we get

$$
\begin{aligned}
0 &= \int \mathrm{d}^3\mathbf{x}\; x_k \left[\frac{\partial(\rho\,\overline{v_i})}{\partial t} + \sum_j \frac{\partial(\rho\,\overline{v_i\,v_j})}{\partial x_j} + \frac{\partial \Phi}{\partial x_i}\,\rho\right] \\
&= \frac{\partial}{\partial t}\int \mathrm{d}^3\mathbf{x}\;\rho\,x_k\,\overline{v_i} - \int \mathrm{d}^3\mathbf{x}\;\rho\,\overline{v_i\,v_k} + \int \mathrm{d}^3\mathbf{x}\;\rho\,x_k\,\frac{\partial \Phi}{\partial x_i}.
\end{aligned}
$$

Defining the tensors $\mathcal{I}_{ik} \equiv \int \mathrm{d}^3\mathbf{x}\;\rho\,x_i\,x_k$ (related to the moment of inertia), $\mathcal{K}_{ik} \equiv \frac{1}{2}\int \mathrm{d}^3\mathbf{x}\;\rho\,\overline{v_i\,v_k}$ (kinetic energy) and $\mathcal{W}_{ik} \equiv -\int \mathrm{d}^3\mathbf{x}\;\rho\,x_k\,\frac{\partial\Phi}{\partial x_i}$ (potential energy), we obtain the tensor virial theorem: $\dfrac{1}{2}\dfrac{\mathrm{d}^2\,\mathcal{I}_{ik}}{\mathrm{d}t^2} = 2\,\mathcal{K}_{ik} + \mathcal{W}_{ik}$.

Its trace gives the more familiar scalar virial theorem $\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}t^2}I = 2K + W$, and in a steady-state system (in "virial equilibrium"), the virial ratio $-2K/W \approx 1$.

This gives the simplest way of roughly estimating the *total* mass of a stellar system: $K = M\overline{v^2},\; W \simeq -G\,M^2/\overline{r} \;\Rightarrow\; M \simeq \overline{v^2}\,\overline{r}/G$ to within a factor of a few.

Summary

- Collisionless Boltzmann equation adequately describes the evolution of large-N stellar systems (at least over short enough time intervals).
- In a steady state, Jeans theorem proclaims that stars should be phase-mixed and their DF may depend only on the integrals of motion.
- Jeans equations offer a simplified description of a stellar system in terms of low-order DF moments.
- They are always valid, but are not sufficient to unambiguously specify the stellar distribution: one needs to make further assumptions, e.g., about the anisotropy and the orientation of the velocity ellipsoid.
- Not every possible solution of Jeans equations corresponds to a physically possible DF.