Modern stellar dynamics, lecture 9:

stellar encounters and collisional evolution

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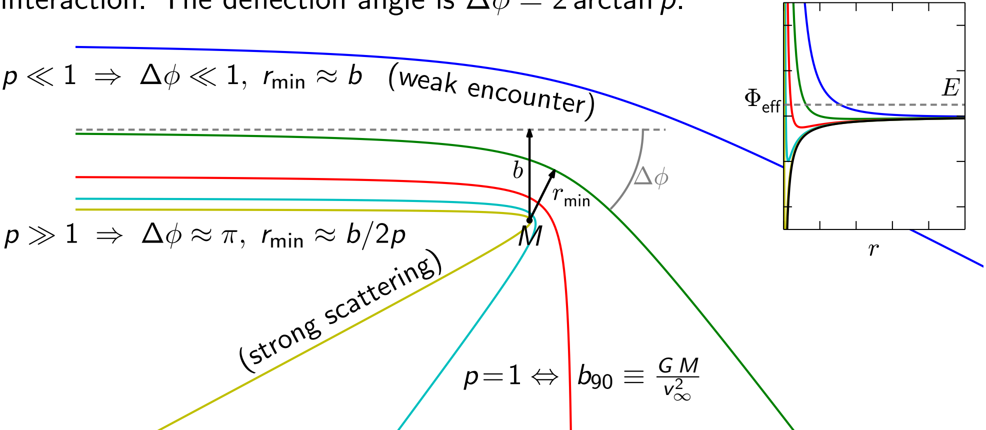
Hyperbolic orbit in a Kepler potential

Recall the general derivation of planar axisymmetric orbits from Lecture 4.
Two integrals of motion $E = \frac{1}{2} v_\infty^2, \quad L = v_\infty \, b$ (b is the impact parameter).
Radial velocity $v_r = \pm\sqrt{2[E - \Phi(r)] - L^2/r^2} = \pm\sqrt{v_\infty^2 \, (1 - b^2/r^2) + 2\, G\, M/r}$.
The distance of closest approach $r_{\min}$ is where $v_r = 0$; solving this quadratic equation, we get $r_{\min} = \frac{b}{p+\sqrt{p^2+1}}$, where $p \equiv \frac{G\,M}{v_\infty^2\, b}$ describes the strength of interaction. The deflection angle is $\Delta\phi = 2 \arctan p$.



Dynamical friction

Now let the mass M move with velocity v through a uniform population of stationary field stars with masses $m_\mathrm{f} \ll M$ and number density n.

If we consider the problem in the moving reference frame associated with the mass M, it is equivalent to the previous setup.

Field stars arrive with all possible impact parameters b; the rate of encounters in the interval $[b \,..\, b+\delta b]$ per unit time is $\delta\nu = 2\pi\, b\, \delta b\, n\, v$. The velocity component of incoming field stars parallel to the motion of the mass M is reduced by

$$\Delta v_\parallel = (1-\cos\Delta\phi)\, v = 2v\,\frac{p^2}{p^2+1} = \frac{2v}{1+(b/b_{90})^2}, \quad \text{where } b_{90} \equiv \frac{G\,M}{v^2}.$$

The conservation of linear momentum implies that the mass M is decelerated: $M\,\dot v = -m_f\, \Delta v_\parallel\, \delta\nu$. Integrating over impact parameters, we get

$$a_\mathrm{DF} = -\int_0^{b_\mathrm{max}} \mathrm{d}b\; 2\pi\, b\; n\, v\, \frac{m_\mathrm{f}}{M}\, \frac{2v}{1+(b/b_{90})^2} = -4\pi\, n\, \frac{m_\mathrm{f}}{M}\, v^2\, b_{90}^2 \ln\sqrt{\cancel{1} + \left[\tfrac{b_\mathrm{max}}{b_{90}}\right]^2} \gg 1$$

$$= -4\pi\, G^2\, \rho_\mathrm{f}\, M\, \ln\Lambda / v^2, \qquad \text{where}$$

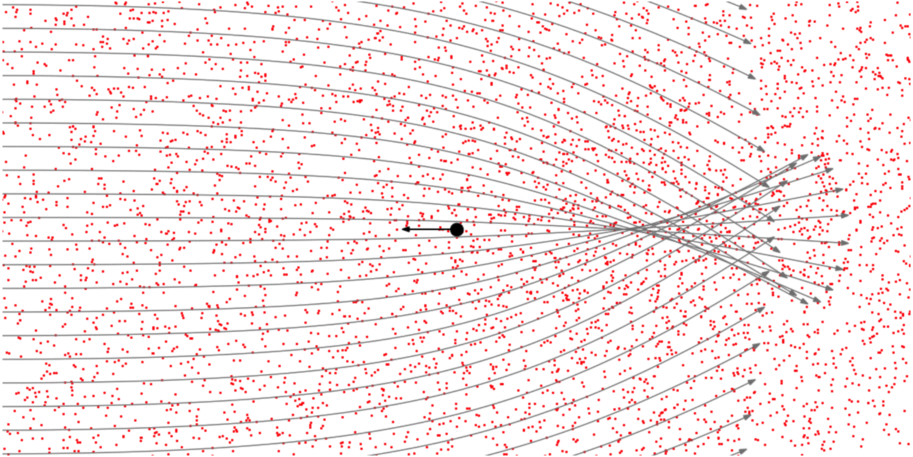
$\rho_\mathrm{f} \equiv n\, m_\mathrm{f}$ is the mass density of field stars, and $\ln\Lambda = \ln(b_\mathrm{max}/b_{90})$ is the Coulomb logarithm. We have to put *some* upper limit b_max, which is usually taken to be the size of the stellar system, or the size of the orbit of the massive object M.

Dynamical friction [Chandrasekhar 1943]

When the field stars have a Maxwellian velocity distribution with 1d dispersion σ,

$$a_{\mathrm{DF}} = -\frac{4\pi\, G^2\, M\, \rho\, \ln\Lambda}{v^2}\left[\operatorname{erf}(X) - \frac{2X}{\sqrt{\pi}}\exp(-X^2)\right], \quad X \equiv \frac{v}{\sqrt{2}\,\sigma}.$$

The dynamical friction can be thought of as a gravitational pull caused by an overdensity of field stars behind the moving mass M [Mulder 1983]:



Dynamical friction properties

– the *acceleration* is proportional to the mass M – more massive objects create a larger density wake and are slowed down more efficiently.

– the acceleration is independent of the masses m_f of field stars (only on their total mass density ρ_f); in this sense, it is a collisionless effect that remains in place even when $m_\mathrm{f} \to 0$.

– the acceleration is caused [primarily] by stars moving slower than the massive body (in a more accurate treatment with a velocity-dependent Coulomb logarithm, some acceleration is also caused by faster stars, but their contribution is smaller by a factor $\ln\Lambda$ [Antonini & Merritt 2012]).

– when the mass M is not much larger than the mass of field stars m_f, a slightly more complicated derivation leads to a similar expression, but with M replaced by $M + m_\mathrm{f}$.

– if the massive object is extended, the lower bound on the impact parameter b_{90} in the Coulomb logarithm is replaced by the characteristic size of the object.

Dynamical friction is usually unimportant for individual stars (although it causes mass segregation in globular clusters, as we will see later), but may lead to a significant orbital decay of globular clusters or satellite galaxies in the host galaxy's potential.

Two-body encounters

Consider now the changes in the perpendicular component of velocity $\Delta v_\perp = v\,\sin\Delta\phi = 2v\,p/(1+p^2)$. Since incoming stars are deflected in all possible directions, the mean change $\langle \Delta v_\perp \rangle = 0$, but the mean-square change is nonzero. Assuming uncorrelated encounters and integrating over all impact parameters, we get the rate of change per unit time

$$\langle \Delta v_\perp^2 \rangle = \int_0^{b_\mathrm{max}} \mathrm{d}b\, 2\pi\, b\, n\, v\, \frac{4v^2\,[b_{90}/b]^2}{(1+[b_{90}/b]^2)^2} \approx \frac{8\pi\, n\, G^2\, M^2\, \ln\Lambda}{v}.$$

The mean-square change in the parallel velocity component $\langle \Delta v_\parallel^2 \rangle$ is the same.

If the field stars have some velocity distribution and their masses are comparable to the test star mass, we need to consider the scattering process in the centre-of-mass frame and transform the velocity changes back to the inertial frame. In the end,

$$\langle \Delta v_\parallel \rangle = -\frac{4\pi\, G^2\,(M+m_\mathrm{f})\,\rho_\mathrm{f}\,\ln\Lambda}{v^2}\,F_1(X), \quad \langle \Delta v_\parallel^2 \rangle = \frac{8\pi\, G^2\, m_\mathrm{f}\,\rho_\mathrm{f}\,\ln\Lambda}{v}\,F_2(X),$$

$$\langle \Delta v_\perp^2 \rangle = \frac{8\pi\, G^2\, m_\mathrm{f}\,\rho_\mathrm{f}\,\ln\Lambda}{v}\,F_3(X),$$ where the dimensionless functions $F_{\dots} \sim \mathcal{O}(1)$ depend on the ratio $X = v/\sqrt{2}\sigma$ and the details of the velocity distribution of field stars [Rosenbluth+ 1957; eq.L.26 in Binney&Tremaine].

Two-body relaxation rate

The time needed to change v^2 by order of itself is $\dfrac{v^2}{\langle \Delta v_\parallel^2 \rangle} = \dfrac{v^3}{8\pi\, G^2\, m_\mathrm{f}\, \rho_\mathrm{f}\, \ln\Lambda}$.

After some futher twiddling, we obtain the relaxation time

$$T_\mathrm{rel} = \frac{0.34\,\sigma^3}{G^2\,\rho_\mathrm{f}\,m_\mathrm{f}\,\ln\Lambda}.$$

One may compare it to the crossing time (orbital period) $T_\mathrm{cross} \simeq L/\sigma$, where L is the characteristic size of the system. From the virial theorem, $G\,M_\mathrm{total} \simeq \sigma^2\,L$, where $M_\mathrm{total} \equiv N\,m_\mathrm{f}$ is the total mass of the system. Thus

$$\frac{T_\mathrm{rel}}{T_\mathrm{cross}} \simeq \frac{0.34\,\sigma^3}{G^2\,\rho_\mathrm{f}\,m_\mathrm{f}\,\ln\Lambda}\frac{\sigma}{L} \simeq \frac{0.34\,(G\,M_\mathrm{total})^2}{G^2\,\rho_\mathrm{f}\,L^3\,m_\mathrm{f}\,\ln\Lambda} \simeq \frac{0.1\,N}{\ln\Lambda},$$

and the Coulomb logarithm is $\ln\Lambda = \ln[L/b_{90}] \simeq \ln[L\,\sigma^2/(G\,m_\mathrm{f})] \simeq \ln N$.

This confirms that in stellar systems with $N \gg 1$, the relaxation time is much longer than the dynamical time (and often longer than the age of the Universe), so they can be meaningfully described in the collisionless approximation for $t \ll T_\mathrm{rel}$.

The above estimates were rather qualitative and careless about numerical factors of order unity; later we will develop a more rigorous theory.

Collisional Boltzmann equation

Denote the 6d phase-space point as $\mathbf{w} \equiv \{\mathbf{x}, \mathbf{v}\}$ or some other canonical coordinates. We start from the same Boltzmann equation for the one-particle distribution function $f(\mathbf{w}, t)$ as in Lecture 6, but now put an *encounter operator* $\Gamma[f]$ in the right-hand side:

$\dfrac{\mathrm{d}f(\mathbf{w},t)}{\mathrm{d}t} \equiv \dfrac{\partial f}{\partial t} + \big[H(\mathbf{w},t), f\big] = \Gamma[f]$, where $[H, f]$ is the Poisson bracket,

$\Gamma[f] = \int \mathrm{d}^6(\Delta\mathbf{w}) \big[\Psi(\mathbf{w}-\Delta\mathbf{w}, \Delta\mathbf{w})\, f(\mathbf{w}-\Delta\mathbf{w}) \;-\; \Psi(\mathbf{w}, \Delta\mathbf{w})\, f(\mathbf{w})\big]$

describes the rate of change of the DF due to stellar encounters,

$\Psi(\mathbf{w}, \Delta\mathbf{w})\, \mathrm{d}^6(\Delta\mathbf{w})\, \Delta t$ is the probability that a star at point $\mathbf{w}$ changes its phase-space coordinates by $\Delta\mathbf{w}$ in a short interval of time Δt.

The two terms in the encounter operator correspond to flux of stars from other points *into* $\mathbf{w}$, and the opposite flux of stars *away* from $\mathbf{w}$, which are proportional to the transition probability times the values of the DF at the points.

So far this is not very constructive; we now make two simplifying assumptions:

– encounters are *weak*, thus Ψ can be expanded to second order in $\Delta\mathbf{w} \ll \mathbf{w}$;

– encounters are *local*, i.e. affect only velocity, but not position (i.e. $\Delta\mathbf{x} = 0$).

Fokker–Planck equation

With these two approximations, the encounter term becomes

$$\Gamma[f] = \int \mathrm{d}^3(\Delta\mathbf{v}) \Big[\Psi(\mathbf{w}-\Delta\mathbf{v}, \Delta\mathbf{v})\, f(\mathbf{w}-\Delta\mathbf{v}) \;-\; \Psi(\mathbf{w}, \Delta\mathbf{v})\, f(\mathbf{w}) \Big]$$

$$\approx \int \mathrm{d}^3(\Delta\mathbf{v}) \Big[\cancel{\Psi(\mathbf{w}, \Delta\mathbf{v})\, f(\mathbf{w})} - \underbrace{\frac{\partial[\Psi\, f]}{\partial\mathbf{v}}}_{\text{gradient (row-vector)}} \Delta\mathbf{v} + \tfrac{1}{2} \Delta\mathbf{v}^T \underbrace{\frac{\partial^2[\Psi\, f]}{\partial\mathbf{v}\partial\mathbf{v}}}_{\text{hessian (tensor)}} \Delta\mathbf{v} \;-\; \cancel{\Psi(\mathbf{w}, \Delta\mathbf{v})\, f(\mathbf{w})} \Big]$$

$$= -\sum_{i=1}^{3} \frac{\partial\big[\langle\Delta v_i\rangle\, f\big]}{\partial v_i} + \frac{1}{2} \sum_{i,j=1}^{3} \frac{\partial^2\big[\langle\Delta v_i \Delta v_j\rangle\, f\big]}{\partial v_i\, \partial v_j}, \quad \text{where}$$

$\langle\Delta v_i\rangle \equiv \int \mathrm{d}^3(\Delta\mathbf{v})\; \Psi(\mathbf{w}, \Delta\mathbf{v})\, \Delta v_i$ and $\langle\Delta v_i \Delta v_j\rangle \equiv \int \mathrm{d}^3(\Delta\mathbf{v})\; \Psi(\mathbf{w}, \Delta\mathbf{v})\, \Delta v_i\, \Delta v_j$ are the advection (drift) and diffusion coefficients evaluated at point $\mathbf{w}$.

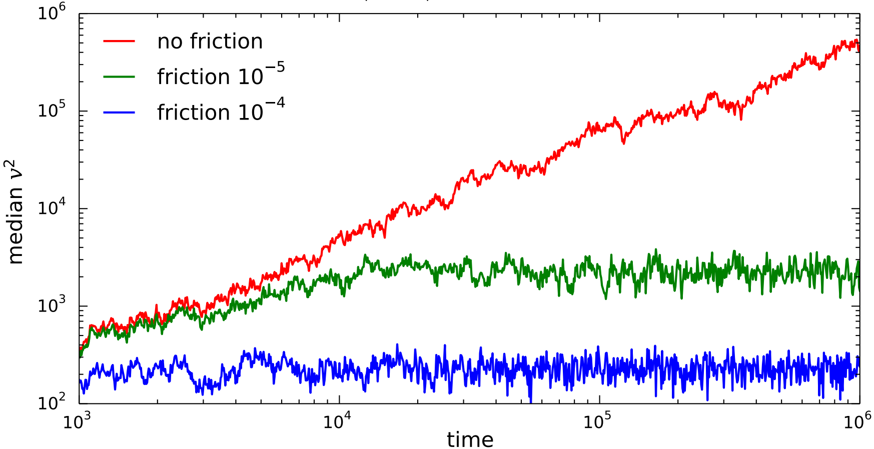
The equation $\mathrm{d}f/\mathrm{d}t = \Gamma[f]$ with the above derived encounter term is known as the Fokker–Planck equation, and is essentially a diffusion equation in velocity space.
It is a *linear* PDE for the DF of *test* stars f (which experience two-body scattering), and the drift and diffusion coefficients are determined by the distribution of *field* stars (which act as scatterers).
However, ultimately the population of field stars consist of all species of test stars.

Fluctuation–dissipation balance

In the ordinary diffusion process, the mean-square velocity would increase linearly with time (random walk). The "purpose" of the drift term is to prevent this, so that v^2 has a finite and time-independent expectation value.

Since $\langle \Delta v \rangle \propto M + m_f$ (as in dynamical friction), the mean square velocity is lower for massive objects ($v^2 \propto M^{-1}$, i.e. equipartition of kinetic energy).

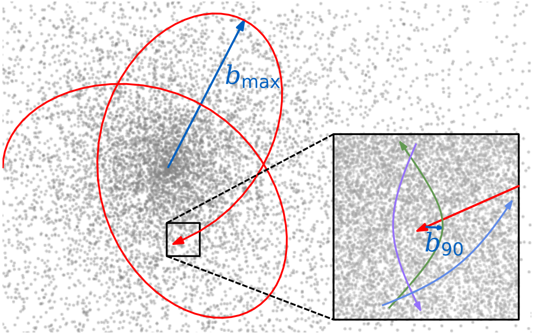
By contrast, diffusion coefficients $\langle \Delta v^2 \rangle$ are independent of M, but $\propto m_f$.



Orbit-averaged Fokker–Planck equation

As we have established that the relaxation time is typically much longer than dynamical time, it makes sense to average the FP equation over the orbit of the test star. This is most clearly expressed by changing variables from position–velocity to action–angle, and then averaging the diffusion coefficients over angles.

In doing so, we make another approximation: even though the cumulative effect of two-body interactions was integrated over impact parameters assuming a spatially uniform distribution of stars moving along straight lines, we now return to a more physically meaningful setup of bound motion in a non-uniform system.



The reason for validity of this approximation is again $\ln\Lambda \sim \mathcal{O}(10) \gg 1$: every logarithmic interval in impact parameter b contributes roughly equally to the relaxation rate, so most of the scattering effect is accumulated in the range $b_{90} \ll b \ll b_{\mathrm{max}}$, where the distribution of field stars is still close to uniform.

Orbit-averaged Fokker–Planck equation

In the action–angle variables, the Fokker–Planck equation takes on a particularly simple form:

$$\frac{\mathrm{d}f(\mathbf{J},t)}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \cancel{\frac{\partial H}{\partial \boldsymbol{\theta}}}\,\frac{\partial f}{\partial \mathbf{J}} - \frac{\partial H}{\partial \mathbf{J}}\,\cancel{\frac{\partial f}{\partial \boldsymbol{\theta}}} = -\sum_{i=1}^{3} \frac{\partial\left[\widetilde{\langle \Delta J_i \rangle}\, f\right]}{\partial J_i} + \tfrac{1}{2}\sum_{i,j=1}^{3} \frac{\partial^2\left[\widetilde{\langle \Delta J_i\, \Delta J_j \rangle}\, f\right]}{\partial J_i\,\partial J_j},$$

where the tilded quantities denote the orbit-averaged advection and diffusion coefficients: $\widetilde{\langle X \rangle} \equiv (2\pi)^{-3} \int \mathrm{d}^3\boldsymbol{\theta}\ \langle X \rangle$.

Often it is more convenient to write down the DF and the FP equation in terms of some other integrals of motion $\mathcal{I}$, e.g., E, L or just E, and recast it in a flux-conservative form:

$$\frac{\partial f}{\partial t} = -\sum\nolimits_i \frac{\partial \mathcal{F}_i}{\partial \mathcal{I}_i}, \quad -\mathcal{F}_i \equiv m_\star\, \mathcal{A}_i\, f + \sum\nolimits_j \mathcal{D}_{ij} \frac{\partial f}{\partial \mathcal{I}_j}.$$

This underlines the fact that the FP equation conserves the total mass of the DF, only redistributing it differently across the integral space.

The coefficients $\mathcal{A}$ and $\mathcal{D}$ are related in a straightforward way to the quantites $\widetilde{\langle X \rangle}$ defined above, and $m_\star$ denotes the mass of test stars.

Orbit-averaged Fokker–Planck equation

In a multi-component system (e.g., light and heavy stars), the DF of each species $f^{(c)}$ satisfies its own FP equation:

$$\frac{\partial f^{(c)}}{\partial t} = -\sum_i \frac{\partial \mathcal{F}_i^{(c)}}{\partial \mathcal{I}_i}, \quad -\mathcal{F}_i^{(c)} \equiv m_\star^{(c)}\, \mathcal{A}_i\, f^{(c)} + \sum_j \mathcal{D}_{ij} \frac{\partial f^{(c)}}{\partial \mathcal{I}_j}.$$

The general derivation of coefficients $\mathcal{A}$ and $\mathcal{D}$ is given in Chapter 5 of the book "Dynamics and evolution of galactic nuclei" [Merritt 2013]. Instead of giving explicit expressions here, we stress the main properties of these coefficients.

Both $\mathcal{A}$ and $\mathcal{D}$ depend on $\mathcal{I}$, contain integrals over DFs of all species weighted by some functions $\mathcal{K}(\mathcal{I})$ that depend on the potential, and are shared between all species.

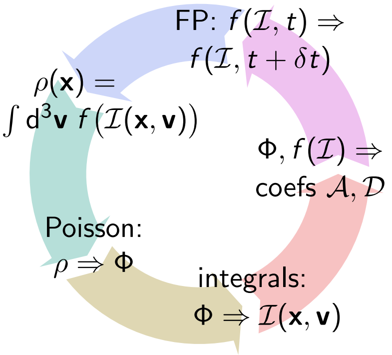
$\mathcal{A}(\mathcal{I}) = \sum_c \int \mathrm{d}\mathcal{I}'\ f^{(c)}(\mathcal{I}')\, \mathcal{K}_\mathcal{A}(\mathcal{I}')$, where $f^{(c)}$ is the mass-normalized DF of species c (i.e., the integral of $f^{(c)}$ over the entire space is the total mass of stars of this kind). Thus $\mathcal{A}$, responsible for dynamical friction, is proportional to the total mass of all stars and doesn't care about the stellar mass, but in the FP equation it is further multiplied by the stellar mass of c-th species (i.e., heavier stars experience stronger friction).

$\mathcal{D}(\mathcal{I}) = \sum_c m_c \int \mathrm{d}\mathcal{I}'\ f^{(c)}(\mathcal{I}')\, \mathcal{K}_\mathcal{D}(\mathcal{I}')$, thus heavier field stars contribute more strongly to the total diffusion (heating) rate, but all species experience the same heating.

Orbit-averaged Fokker–Planck equation

Although it's not immediately obvious, the FP equation is also energy-conservative. However, unlike mass conservation (which is local in the space of integrals), the energy exchange is mediated by the coefficients $\mathcal{A}$, $\mathcal{D}$ across the entire energy space (non-locally). In other words, two stars with energies E_1 and E_2 experience a scattering interaction and exchange a tiny bit of energy δE (in the FP approximation) and therefore shift only slightly in the energy space to $E_1 + \delta E$, $E_2 - \delta E$, but the energy was transferred over a large "gap" $E_2 - E_1 \gg \delta E$.

To study the time evolution of a stellar system under two-body relaxation, the FP equation needs to be complemented by the relation between the DF $f(\mathcal{I})$ and the density $\rho(\mathbf{x})$, by the Poisson equation linking ρ to Φ, by the expressions for the integrals $\mathcal{I}(\mathbf{x}, \mathbf{v};\ \Phi)$, and by the coefficients $\mathcal{A}, \mathcal{D}$.
This system of integro-differential equations may be solved numerically by finite differences on grids in $\mathcal{I}$, $\mathbf{x}$, or by Monte Carlo methods.



Summary

- Collisional effects are "weak" if $N \gg 1$, equivalently $T_{\rm rel} \gg T_{\rm dyn}$.
- The "classical" Chandrasekhar theory of two-body relaxation is developed for an idealized infinite homogeneous system, but due to the long-range nature of gravity, is logarithmically divergent at large impact parameters. In the analogous theory for Coulomb interactions in plasma, the upper cutoff $b_{\rm max}$ corresponds to the Debye screening radius, but in gravity, all charges have the same sign and there is no screening.
- The Fokker–Planck approximation further neglects strong interactions with $b \lesssim b_{90}$.
- Despite obvious internal inconsistencies, this theory adequately works in practice thanks to the (moderately) large value of the Coulomb logarithm $\ln\Lambda \sim \mathcal{O}(10)$.
  Its predictions are validated by N-body simulations and are correct at a level of $10-20\%$ (after a suitable tuning of the Coulomb logarithm).
- A more rigorous kinetic theory of inhomogeneous self-gravitating systems is *considerably* more complicated, both conceptually and in practice.