Modern stellar dynamics – problem sheet 1

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Examples class: Thu 17 Feb, 13:30, MR3

Problem 1.1: projection of ellipsoidal bodies.

(i) Consider an axisymmetric density profile $\rho(\boldsymbol{r}) = \rho(m)$, where $m \equiv \sqrt{x^2 + y^2 + (z/q)^2}$ is the ellipsoidal radius, q is the axis ratio ($q < 1$ for oblate and $q > 1$ for prolate models), and $\rho(m)$ is an arbitrary function. Let i be the angle between the line of sight and the z axis of the model (usually called the inclination angle for oblate models). Derive a formula for the projected axis ratio q' as a function of q and i for both oblate and prolate models.

(ii) Consider now a triaxial ellipsoidal model with axis ratios $p = y/x$, $q = z/x$; $1 \ge p \ge q$. Can it appear round in projection ($q' = 1$) for any choice of p and q? If yes, determine the orientation angles that would produce a round projection; if no, give a counter-example.

Problem 1.2: potential of flattened systems.

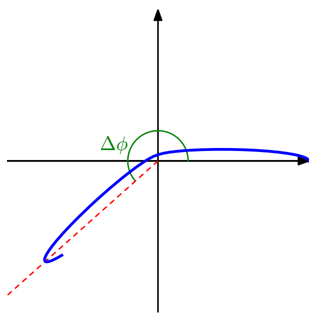
(i) Consider the singular isothermal (logarithmic) potential $\Phi(m) = \Phi_0 \ln(m/r_0)$, where $m \equiv \sqrt{x^2 + y^2 + (z/q_\mathrm{p})^2}$ is the ellipsoidal radius and $q_\mathrm{p} \le 1$ is the potential axis ratio. Determine the corresponding density profile $\rho(R, z)$ and compute its axis ratio q_d defined by the condition $\rho(R, 0) = \rho(0, q_\mathrm{d} R)$. In what range of q_p is the resulting density non-negative everywhere in space?

(ii) Consider a flattened power-law density profile $\rho(m) = \rho_0\, (m/r_0)^{-\gamma}$ with $2 < \gamma < 3$, where $m \equiv \sqrt{x^2 + y^2 + (z/q_\mathrm{d})^2}$. In the limit of small flattening ($q_\mathrm{d} \to 1$), compute the corresponding potential in the spherical-harmonic approximation up to $\ell = 2$ and determine the axis ratio of the equipotential surfaces q_p. How does it compare with the previous result? Explain qualitatively the behaviour of the resulting expression when $\gamma \to 3$.

Note: Legendre polynomials are defined as $P_\ell(x) \equiv \dfrac{1}{2^\ell\, \ell!} \dfrac{\mathrm{d}^\ell}{\mathrm{d}x^\ell} (x^2 - 1)^\ell$.

Problem 1.3: radial and azimuthal periods.

Consider a spherical power-law potential $\Phi(r) = \Phi_0\, (r/r_0)^{2-\gamma}$ with $\Phi_0 > 0$ for $0 \le \gamma < 2$ or $\Phi_0 < 0$ for $2 < \gamma \le 3$. Determine the angle $\Delta\phi$ between two successive apocentres (see figure) of a nearly-radial orbit (in the limit $L \to 0$). Hint: consider the cases $\gamma > 2$ and $\gamma < 2$ separately.



Determine the ratio between radial and azimuthal periods T_r/T_ϕ for a nearly-radial orbit. Compare with the same ratio for a nearly-circular orbit, using the epicyclic frequencies.

You may use the Beta function $\mathrm{B}(a, b) = \displaystyle\int_0^1 \mathrm{d}x\; x^{a-1}\, (1-x)^{b-1}$; $\quad \mathrm{B}(a,\, 1-a) = \dfrac{\pi}{\sin \pi a}$.

Problem 1.4: potential and peri/apocentre radii.

(i) Show that the potential of a spherical system with non-negative density is monotonic with radius. Does this hold in a more general geometry? If not, provide a counter-example.

(ii) Write down the equation defining the turning points (pericentre and apocentre radii) in a spherical potential generated by a non-negative density distribution, and prove that it may have at most two solutions. Hint: use a substitution $u = 1/r$.

MODERN STELLAR DYNAMICS – PROBLEM SHEET 2

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Examples class: Thu 3 Mar, 13:30, MR3

Problem 2.1: action in a generic 1d potential.

Consider a one-dimensional system described by a Hamiltonian $H(x,v) = \Phi(x) + \frac{1}{2}v^2$, where $\Phi(x)$ is an arbitrary potential that is monotonic in x for $x \ge 0$, symmetric w.r.t. change of sign of x and is zero at origin. The motion is confined in the interval $-x_{\max}(E)..x_{\max}(E)$, where $x_{\max}$ is the inverse function for the potential. Since the Hamiltonian is an even function of both x and v, we can compute the action by integrating over one quadrant:

$$J = \frac{2}{\pi}\int_0^{x_{\max}(E)} \mathrm{d}x\, \sqrt{2\big[E - \Phi(x)\big]}.$$

There is a one-to-one correspondence between J and E.

(i) Now imagine that you know $H(J)$. Determine the potential $\Phi(x)$.

Hint: use the Abel integral inversion to determine $x_{\max}$ as a function of energy, then the potential is the inverse function $x_{\max}^{-1}(\Phi)$.

(ii) Verify the derivation for the case of simple harmonic oscillator.

(iii) Determine the potential for the case $H(J) = c\,J^{4/3}$.

You may use the Beta function $\mathrm{B}(a,b) = \displaystyle\int_0^1 \mathrm{d}x\, x^{a-1}\,(1-x)^{b-1}$; $\quad \mathrm{B}(a,\,1-a) = \dfrac{\pi}{\sin \pi a}$.

Problem 2.2: stellar distribution around a massive black hole.

Galactic nuclei often contain central supermassive black holes, which dominate the gravitational potential in their vicinity. Consider a stellar cluster with a power-law density profile $\rho = \rho_0\,(r/r_0)^{-\gamma}$ in the spatial region where the potential is $\Phi(r) = -G\,M_\bullet/r$ (i.e., at radii where the enclosed stellar mass is much smaller than $M_\bullet$).

(i) Determine the isotropic DF $f(E)$ that generates the required density profile. For which values of γ is such a DF possible?

(ii) Consider a spherical DF of the form $f(E,L) = f_0\,L^{-2\beta}\,|E|^n$. Show that it produces the required density profile for a suitable combination of β and n; find the range of values for which it is possible. Compute the radial and tangential velocity dispersions and show that the velocity anisotropy coefficient is β.

(iii) Solve the spherical Jeans equation with a constant anisotropy coefficient β for the given density and potential and determine the radial velocity dispersion. Compare it with the result of the previous calculation. What is the permitted range of γ and β in the Jeans solution? Does it differ from the DF-based result, and if yes, why?

Problem 2.3: vertically isothermal disc profile.

The vertical structure of a thin stellar disc can be studied in a one-dimensional approximation (neglecting the radial gradients), writing the density and potential as functions of z. Consider an isothermal DF of stars

$$f(z, v_z) = \frac{\rho_0}{\sqrt{2\pi}\,\sigma}\exp(-E_z/\sigma^2), \quad \text{where } E_z = \Phi(z) + v_z^2/2.$$

(i) Show that the gravitational potential of the form $\Phi(z) = \Phi_0 \ln\cosh(z/h)$ and the corresponding density are consistent with this DF for a suitably chosen normalization Φ_0 and scale height h; determine Φ_0 and h.

(ii) Compute the surface density of this model and the fraction of stars with $|z| > h$.

Problem 2.4: two-integral axisymmetric model for a flattened logarithmic potential.

Consider the flattened logarithmic potential model (same as in Problem 1.2):
$\Phi(R, z) = \Phi_0\ \ln(m/r_0), \quad m \equiv \sqrt{R^2 + (z/q)^2}$.

(i) Write down the corresponding density profile and the circular velocity in the $z = 0$ plane.

(ii) Show that this potential–density pair can be produced by a very simple two-integral DF of the form
$f(E, L_z) = A\ \exp(-2E/\Phi_0) + B\, L_z^2\ \exp(-4E/\Phi_0)$,
determine the constants A and B.

(iii) Compute the second moments of velocity in cylindrical coordinates $\overline{v^2_{R,z,\phi}}$ and show that two of them are equal.

Modern stellar dynamics – problem sheet 3

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Examples class: Thu 17 Mar, 13:30, MR3

Problem 3.1: dynamical friction in the Fornax dwarf galaxy.

Fornax dSph is one of the most massive Milky Way satellites, and contains an unusually high number of globular clusters for its luminosity (6). Despite being formed more than 10 Gyr ago, they escaped the fate of sinking to the galaxy centre via dynamical friction. This places interesting constraints on the mass distribution in the galaxy: namely, several studies argued that it could not have a cusped density profile, otherwise the dynamical friction would be too efficient.

(i) Consider a singular isothermal sphere profile $\rho = \rho_0\,(r/r_0)^{-2}$. Derive the corresponding circular velocity $v_{\rm circ}$, and from the spherical isotropic Jeans equation determine the 1d velocity dispersion σ.

The observed line-of-sight velocity dispersion in Fornax is ~ 12 km/s. Assuming that the [total] density profile follows the singular isothermal law, compute the density ρ_0 at $r_0 = 1$ kpc.

(ii) Using the Chandrasekhar dynamical friction formula,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{4\pi\,G^2\,M\,\rho\,\ln\Lambda}{v^2}\left[\mathrm{erf}(X) - \frac{2X}{\sqrt{\pi}}\exp(-X^2)\right], \quad X \equiv v/(\sqrt{2}\,\sigma),$$

determine the evolution of the radius of a circular orbit in a singular isothermal potential as a function of time.

Hint: the friction force reduces energy and angular momentum, but the orbit remains circular – we may write the angular momentum loss $\mathrm{d}L_{\rm circ}/\mathrm{d}t$ as $r\,\mathrm{d}v/\mathrm{d}t$, then put in the dynamical friction acceleration, and then reinterpret the decay rate as $\mathrm{d}(r_{\rm circ}\,v_{\rm circ})/\mathrm{d}t$. With the previously derived expression for $v_{\rm circ}$, this translates to the time derivative of the radius of the orbit. Solve the resulting ODE and find the sinking time.

Compute the sinking time for a $M = 2\times10^5\,M_\odot$ globular cluster in the Fornax galaxy, assuming that it started at a radius 1 kpc and using the Coulomb logarithm $\ln\Lambda = 5$. The expression in square brackets evaluates to 0.20, 0.43, 0.74, 0.95 for $X = 1/\sqrt{2}$, 1, $\sqrt{2}$, 2, and is $\propto X^3$ for small X.

(iii) How would the result change (qualitatively) if the Fornax galaxy instead had a constant-density core? Although the mass distribution in this case is somewhat more difficult to determine (one needs to put a finite outer cutoff radius, otherwise a physically valid solution does not exist), in the first approximation one may assume that the density at 1 kpc and the velocity dispersion are the same as in the cuspy (singular isothermal) case and remain constant at smaller radii.

Problem 3.2: rate of stellar encounters and their effect on planetary systems.

Consider a population of Solar-mass stars with a number density n and 1d velocity dispersion σ. Determine the rate of encounters $\nu(r_{\rm min})$ as a function of the distance of closest approach $r_{\rm min}$, in the regime of weak or strong gravitational focusing.

Estimate the distance of the closest approach with another star that the Solar system might have had in the past 4.5 Gyr, assuming the density and velocity dispersion of stars in the Solar neighbourhood to be $n \simeq 0.05\,\mathrm{pc}^{-3}$, $\sigma = 25$ km/s. Is the closest approach likely to be in the regime of strong focusing? How does $r_{\min}$ compare with the sizes of planetary orbits? The semimajor axis of Pluto is 39 AU, Sedna (one of the most distant dwarf planets beyond Neptune) and the hypothetical Planet Nine have semimajor axes ~ 500 AU. 1 AU $= \pi/(180 * 60 * 60) \approx 5 \times 10^{-6}$ pc.

Repeat the calculation for a star in the core of a dense globular cluster ($n \simeq 10^5\,\mathrm{pc}^{-3}$, $\sigma = 10$ km/s). Should we expect to find Earth-like planets in the habitability zone in globular clusters?

Problem 3.3: relaxation time in the Milky Way nucleus.

The density profile of the Milky Way nuclear star cluster is $\rho(r) \approx \rho_0\,(r/r_0)^{-\gamma}$ with $r_0 = 1$ pc, $\rho_0 = 10^5\,M_\odot/\mathrm{pc}^3$, and $\gamma = 3/2$.

(i) Determine the radius of influence of the supermassive black hole with mass $M_\bullet \approx 4 \times 10^6\,M_\odot$ (defined as the radius containing the stellar mass comparable to $M_\bullet$, or equivalently, the radius within which the gravitational force is dominated by the black hole).

(ii) Using the results of Problem 2.2, determine the 1d velocity dispersion as a function of radius inside the region of influence, assuming isotropic velocity distribution. Hint: despite the caveats of Jeans equations, one may them in this case, since the density profile is steeper than the limiting value $\gamma_{\min} = 1/2$ for the isotropic distribution function.

(iii) Compute the relaxation time, assuming a stellar population composed of $1\,M_\odot$ stars and a Coulomb logarithm $\ln\Lambda = 10$. Should we expect the system to be in thermodynamical equilibrium and have isotropic velocity distribution?

Problem 3.4 Gravitational slingshot.

In a famous sci-fi musical co-written by a Cambridge astronomy student, a starship travels from Earth to a nearby star with velocity $v \ll c$ (say, $v = 10^4$ km/s, although the actual value is not mentioned in the story). However, due to unexpected technical problems, it has to turn back after 15 years in space and needs to return to Earth in only 5 years. It performs a slingshot maneuvre around a neutron star (which just happened to be in the right place at the right time) to achieve this goal.

(i) Is it possible that the return time is $3\times$ shorter than the duration of the outward journey? If yes, what are the additional circumstances and conditions?

(ii) How close to the neutron star does it need to fly to achieve the goal? Would the crew survive the G force (assuming that a trained astronaut can withstand $10g$)? How about the tidal forces?