# Action–angle variables in galactic dynamics

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# Hamiltonian mechanics

Consider a particle moving in a potential $\Phi(\mathbf{x})$.

$\mathbf{x}(t), \mathbf{v}(t)$ are "ordinary" $D$-dimensional position/velocity coordinates;

$H(\mathbf{x}, \mathbf{v}) = \Phi(\mathbf{x}) + \frac{1}{2}|\mathbf{v}|^2$ is the Hamiltonian.

The equations of motion are

$$\frac{d\mathbf{x}}{dt} \equiv \dot{\mathbf{x}} = \mathbf{v}, \quad \frac{d\mathbf{v}}{dt} \equiv \dot{\mathbf{v}} = -\frac{\partial \Phi}{\partial \mathbf{x}}$$

One may consider a general class of Hamiltonian systems defined by $H(\mathbf{q}, \mathbf{p})$ as a function of generalized phase-space coordinates, which satisfy the Hamilton's equations of motion:

$$\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}, \qquad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}$$

# Poisson brackets

Define the commutator operator for two functions of phase-space coordinates $A(\mathbf{q},\mathbf{p})$ and $B(\mathbf{q},\mathbf{p})$ as

$$[A,B] \equiv \frac{\partial A}{\partial \mathbf{q}}\frac{\partial B}{\partial \mathbf{p}} - \frac{\partial A}{\partial \mathbf{p}}\frac{\partial B}{\partial \mathbf{q}}.$$

It follows immediately that

$$[A,A] = 0, \quad [A,B] = -[B,A], \qquad \text{(antisymmetry)}$$

$$\big[[A,B],C\big] + \big[[B,C],A\big] + \big[[C,A],B\big] = 0, \qquad \text{(Jacobi identity)}$$

$$[q_i,q_j] = 0, \quad [p_i,p_j] = 0, \quad [q_i,p_j] = \delta_{ij}, \qquad i,j = 1..D,$$

and the Hamilton equations can be written as

$$\dot{q}_i = [q_i,H], \quad \dot{p}_i = [p_i,H]$$

# Integrals of motion

If $[A, B] = 0$, we say that $A$ commutes with $B$.

If a function $A(\mathbf{q}, \mathbf{p})$ commutes with the Hamiltonian, it is conserved along the particle's trajectory – we call it an integral of motion:

$$\begin{aligned}\frac{dA}{dt} &= \frac{\partial A}{\partial \mathbf{q}}\frac{d\mathbf{q}}{dt} + \frac{\partial A}{\partial \mathbf{p}}\frac{d\mathbf{p}}{dt} \\ &= \frac{\partial A}{\partial \mathbf{q}}\frac{\partial H}{\partial \mathbf{p}} - \frac{\partial A}{\partial \mathbf{p}}\frac{\partial H}{\partial \mathbf{q}} \\ &= [A, H] = 0\end{aligned}$$

Obviously, the Hamiltonian itself is an integral of motion.

Phase-space distribution function $f(\mathbf{q}, \mathbf{p})$ satisfies the collisionless Boltzmann equation and hence is also conserved along the trajectory of any particle.

# Canonical transformations

Consider a change of variables from $\mathbf{p}, \mathbf{q}$ to $\mathbf{P}, \mathbf{Q}$, and express the Hamiltonian $H(\mathbf{P}, \mathbf{Q})$ or any other function in phase space in terms of the new variables.

If the new variables satisfy the canonical commutatation relations
$[Q_i, Q_j] = 0, \quad [P_i, P_j] = 0, \quad [Q_i, P_j] = \delta_{ij},$
such transformation is called canonical (or symplectic).

It also preserves

- Hamilton's equations of motion:
  $\dot{\mathbf{Q}}_i = [Q_i, H], \; \dot{\mathbf{P}}_i = [P_i, H];$
- more generally, all Poisson brackets:
  $\big[A(\mathbf{p}, \mathbf{q}), B(\mathbf{p}, \mathbf{q})\big] = \big[A(\mathbf{P}, \mathbf{Q}), B(\mathbf{P}, \mathbf{Q})\big];$
- all Poincaré invariants: $\oint \mathbf{p} \cdot d\mathbf{q}$
- $2D$-dimensional phase volume element: $d^D\mathbf{q}\, d^D\mathbf{p} = d^D\mathbf{Q}\, d^D\mathbf{P}$

# Examples of canonical transformations

1. Exchange: $\mathbf{Q} = \mathbf{p},\ \mathbf{P} = \mathbf{q}$

   (i.e., there is no fundamental difference between coordinate and momentum variables).

2. Point transformation: define $\mathbf{Q}(\mathbf{q})$ in whatever way, and then $\mathbf{P}(\mathbf{q}, \mathbf{p})$ is uniquely specified.

   For instance, cartesian to polar coordinates: $\mathbf{q} \equiv \{x,\ y\}$ to $\mathbf{Q} \equiv \{r,\ \phi\}$ implies $\mathbf{P} \equiv \{p_r,\ p_\phi\} = \big\{(xp_x + yp_y)/r,\ xp_y - yp_x\big\}$.

3. Hamiltonian flow: integrate the equations of motion for some time $\tau$, and let $\{\mathbf{Q}, \mathbf{P}\}(\mathbf{q}, \mathbf{p}; \tau)$ be the new coordinates and momenta of a point started from initial conditions $\mathbf{q}, \mathbf{p}$.

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One powerful way of constructing such transformations is to introduce a generating function $F(\mathbf{q}, \mathbf{P})$ such that $\mathbf{p} = \dfrac{\partial F}{\partial \mathbf{q}}, \quad \mathbf{Q} = \dfrac{\partial F}{\partial \mathbf{P}}$;

$F$ could also be expressed in terms of some other combination of old and new variables, e.g., $F(\mathbf{q}, \mathbf{Q})$, etc.

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A free particle!

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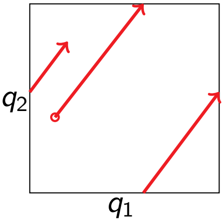
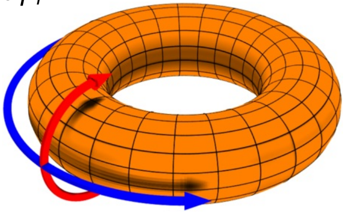
The next simplest (and more realistic) thing? Periodic motion:

$$H(\mathbf{q},\mathbf{p}) = H(\mathbf{p}) \quad \Longrightarrow \quad p_i(t) = \text{const}, \quad q_i(t) = \frac{\partial H}{\partial p_i}\, t + \text{const},$$

where $\mathbf{q}$ are treated as angle-like (periodic) variables, $q + 2\pi \cong q$ on a $D$-dimensional torus.



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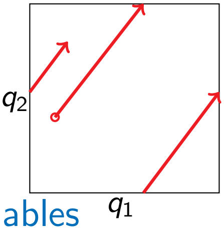
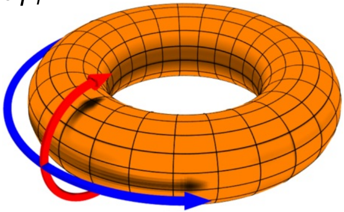


These are action–angle variables

# Integrability and the Arnold–Liouville theorem

If $I_1$ and $I_2$ are two integrals of motion, then the Jacobi identity $\big[[I_1, I_2], H\big] + \big[[I_2, H], I_1\big] + \big[[H, I_1], I_2\big] = 0$ implies that $[I_1, I_2]$ is also an integral of motion.

(Example: $I_1 = L_x, I_2 = L_y \Rightarrow [I_1, I_2] = L_z$).

If $[I_1, I_2]$ is identically zero, the two integrals are said to be in involution.

A Hamiltonian system with $D$ degrees of freedom is *integrable* if it has $D$ independent integrals of motion $I_1 \dots I_D$ (including the Hamiltonian itself) which are all in involution with each other.

The motion of any particle is restricted to a $D$-dimensional hypersurface of the $2D$-dimensional phase space.

**Theorem:** this hypersurface is diffeomorphic to (i.e., could be smoothly deformed into) a $D$-torus, parametrized by $D$ periodic variables $\boldsymbol{\theta} \in [0..2\pi)$.

# Action–angle variables for a 1d simple harmonic oscillator

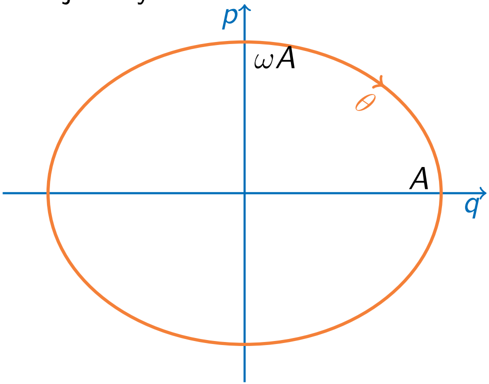
The simplest possible Hamiltonian system: $H(q,p) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$.

The trajectory is $q(t) = A\sin(\omega t + \phi_0)$, $p(t) = A\omega\cos(\omega t + \phi_0)$, and the energy is $E = \frac{1}{2}\omega^2 A^2$.

The motion is periodic with frequency $\omega$ ($\Leftrightarrow$ period $2\pi/\omega$), so we define the angle $\theta = \omega t + \phi_0$.

The action $J$ is $\frac{1}{2\pi}\times$area enclosed by the trajectory:

$$
\begin{aligned}
J &= \frac{1}{2\pi}\oint p\, dq \\
&= \frac{1}{2\pi}\int_0^{2\pi} p(\theta)\,\frac{dq}{d\theta}\,d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} A^2\omega\cos^2\theta\,d\theta \\
&= \frac{A^2\omega}{2} = \frac{E}{\omega}
\end{aligned}
$$



# Action–angle variables for a 2d simple harmonic oscillator

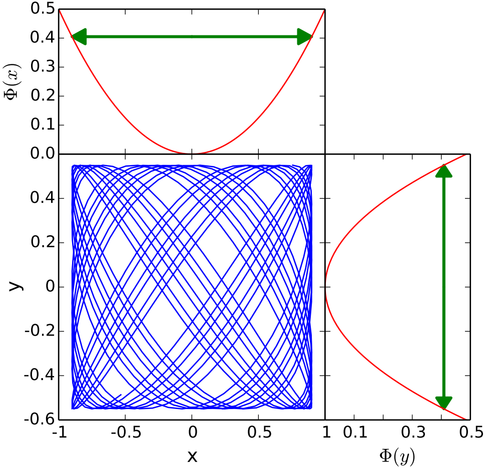
The same thing but in two dimensions: $\mathbf{q} = \{x, y\}$, $\mathbf{p} = \{p_x, p_y\}$;

Hamiltonian:

$$H(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\left(p_x^2 + \omega_x^2 x^2\right) + \frac{1}{2}\left(p_y^2 + \omega_y^2 y^2\right)$$

$$\equiv \quad H_x(x, p_x) \quad + \quad H_y(y, p_y)$$

Motion is separable in $x, y$ –
two uncoupled simple harmonic oscillators,
two integrals of motion $E_x, E_y$,
actions are $J_x = E_x/\omega_x, J_y = E_y/\omega_y$.



# Action–angle variables for a 2d planar axisymmetric potential

A slightly more complicated system: two degrees of freedom, motion in an axisymmetric potential $\Phi(x, y) = \Phi(R)$, where $R \equiv \sqrt{x^2 + y^2}$.

Canonical coordinates: $\mathbf{q} = \{R, \phi\}, \mathbf{p} = \{p_R, p_\phi\}$

$$\text{Hamiltonian: } H = \Phi(R) + \frac{1}{2}\left(p_R^2 + \frac{p_\phi^2}{R^2}\right) \equiv \Phi_{\text{eff}}(R) + \frac{1}{2}p_R^2$$

$$\text{equations of motion: } \dot{R} = p_R, \quad \dot{\phi} = \frac{p_\phi}{R^2}, \quad \dot{p}_R = -\frac{d\Phi_{\text{eff}}}{dR}, \quad \dot{p}_\phi = 0$$

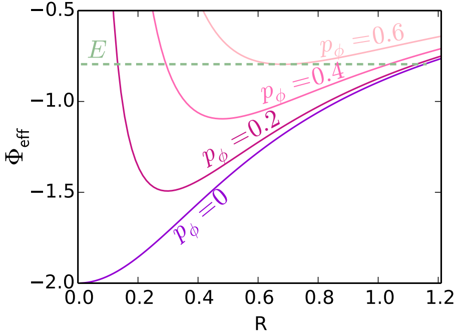
integrals of motion: $E$ and $p_\phi$

Motion in $R$ is described by a 1d effective potential $\Phi_{\text{eff}}(R) \equiv \Phi(R) + p_\phi^2/R^2$

The radial action is

$$J_R = \frac{1}{\pi}\int_{R_-}^{R_+} p_R(R;\, E, p_\phi)\, dR$$

$$= \frac{1}{\pi}\int_{R_-}^{R_+} \sqrt{2\left[(E - \Phi_{\text{eff}}(R)\right]}\, dR$$



# Action–angle variables for a 2d planar axisymmetric potential

Motion in $\phi$: $\dot{p}_\phi = 0 \quad \Rightarrow \quad p_\phi = \text{const},$

hence the azimuthal action is

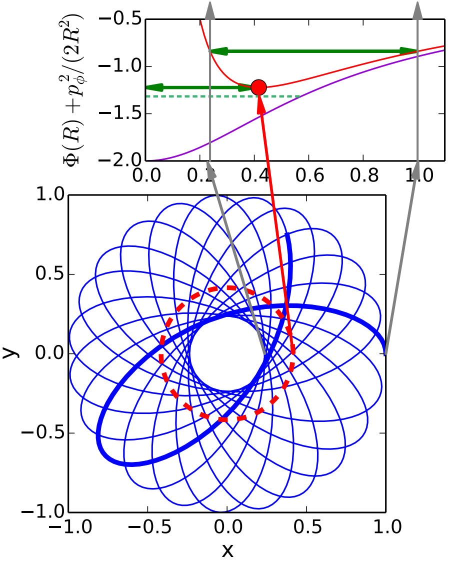
$$J_\phi = \frac{1}{2\pi} \int_0^{2\pi} p_\phi \, d\phi = p_\phi.$$

The actions $J_R, J_\phi$ describe the extent of the orbit in two complementary dimensions:

$J_\phi$ corresponds to the "guiding radius" (the radius of a circular orbit with the given angular momentum $J_\phi$),

$J_R$ gives the extent of radial oscillation about this guiding radius.

They can be varied independently, and any possible choice (provided that $J_R \geq 0$) corresponds to some trajectory.



# Angles and frequencies

Note that $\dot{\phi} = p_\phi / R^2(t) \neq$ const, so $\phi$ is not a canonically conjugate angle variable to $p_\phi$!

Such variable is $\theta_\phi$ defined to increase linearly with time,
and similarly the radial phase angle $\theta_R$ also increases linearly with time:

$\theta_R = \Omega_R\, t, \quad \theta_\phi = \Omega_\phi\, t$, where

$$\Omega_R \equiv \frac{\partial H(J_R, J_\phi)}{\partial J_R}, \quad \Omega_\phi \equiv \frac{\partial H(J_R, J_\phi)}{\partial J_\phi} \quad \text{are orbital frequencies.}$$

$$\theta_R(R;\, E, p_\phi) = \Omega_R \int_{R_-}^{R} \frac{dt}{dR}\, dR = \Omega_R \int_{R_-}^{R} \frac{dR}{p_R(R;\, E, p_\phi)}$$

Radial orbital period $T_R \equiv \dfrac{2\pi}{\Omega_R} = 2 \displaystyle\int_{R_-}^{R_+} \frac{dR}{p_R} = 2 \int_{R_-}^{R_+} \frac{dR}{\sqrt{2\left[E - \Phi(R)\right] - \frac{p_\phi^2}{R^2}}}$

Azimuthal period $T_\phi \equiv \dfrac{2\pi}{\Omega_\phi} = \dfrac{2\pi\ \int_{R_-}^{R_+} dR/p_R}{p_\phi\ \int_{R_-}^{R_+} dR/(R^2\, p_R)}$

# Action–angle variables for a 3d spherical potential

Spherical coordinates: $r, \theta, \phi, p_r, p_\theta, p_\phi$

Hamiltonian: $H = \Phi(r) + \frac{1}{2}\Big(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\,\sin^2\theta}\Big)$

Integrals of motion: $E, L_x, L_y, L_z\ \Big[, L \equiv \sqrt{L_x^2 + L_y^2 + L_z^2}\,\Big]$

Radial action: $J_r = \frac{1}{\pi}\int_{r_-}^{r_+}\sqrt{2\big[E - \Phi(r)\big] - \frac{L^2}{r^2}}$ $\geq 0$

Azimuthal action: $J_\phi = L_z$ (any sign)

Vertical action: $J_\theta \equiv J_z = L - |L_z|$ $\geq 0$

In general, actions, angles, frequencies, or $H(\mathbf{J})$ do not have analytic expressions. One exception is the isochrone potential [Hénon 1959]:

$$\Phi(r) = -\frac{G\,M}{b + \sqrt{b^2 + r^2}}$$ (includes Kepler and harmonic oscliiator as limiting cases)

$$H(\mathbf{J}) = -\frac{2\,(G\,M)^2}{\big(2J_r + L + \sqrt{L^2 + 4\,G\,M\,b}\big)^2}$$

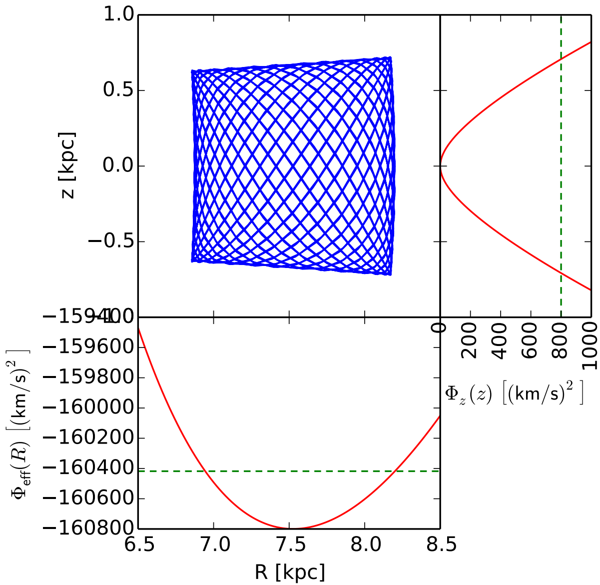
# Action–angle variables for a 3d axisymmetric potential

For nearly-circular orbits close to the equatorial plane, one may use the **epicyclic approximation**:

$$\Phi(R,z) \approx \Phi_R(R) + \Phi_z(z),$$

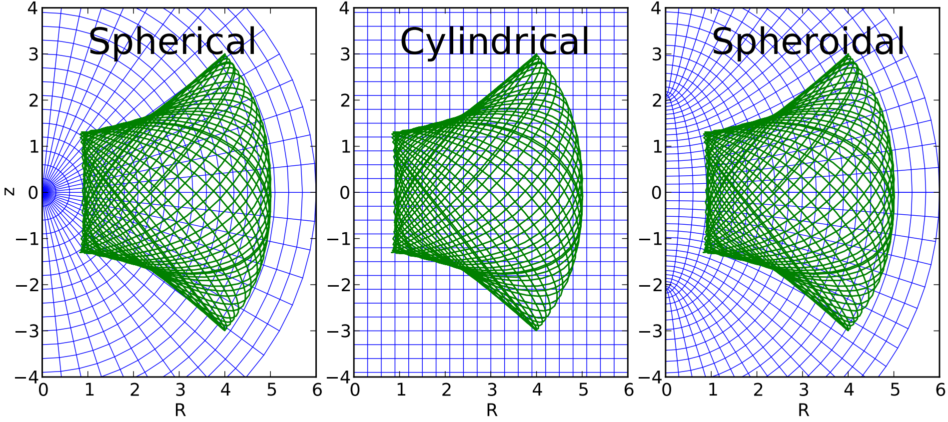
motion in $R, \phi$ as in the planar axisymmetric problem with effective potential $\Phi_{\text{eff}} = \Phi_R(R) + \frac{1}{2}L^2/r^2$, and independent, nearly harmonic motion in $z$.

However, it becomes increasingly inaccurate for orbits with high eccentricity and/or inclination.



# State of the art: Stäckel fudge

Fact: orbits in realistic axisymmetric galactic potentials are much better aligned with prolate spheroidal coordinates.



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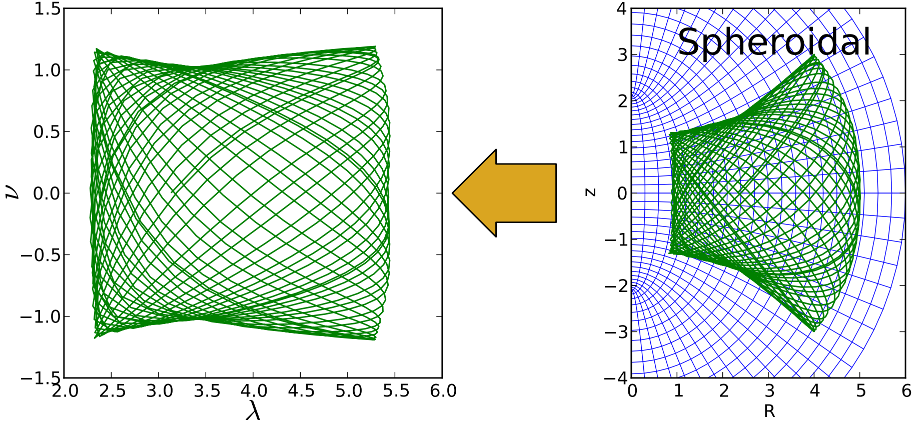
One may explore the assumption that the motion is separable in these coordinates $(\lambda, \nu)$.



## Stäckel fudge [Binney 2012]

The most general form of potential that satisfies the separability condition is the Stäckel potential[1]: $\Phi(\lambda,\nu) = -\dfrac{f_1(\lambda) - f_2(\nu)}{\lambda - \nu}$.

The motion in $\lambda$ and $\nu$ directions, with canonical momenta $p_\lambda, p_\nu$, is governed by two separate equations:

$$2(\lambda - \Delta^2)\,\lambda\,p_\lambda^2 = \left[E - \frac{L_z^2}{2(\lambda - \Delta^2)}\right]\lambda - [I_3 + (\lambda - \nu)\Phi(\lambda,\nu)],$$
$$2(\nu - \Delta^2)\,\nu\,p_\nu^2 = \left[E - \frac{L_z^2}{2(\nu - \Delta^2)}\right]\nu - [I_3 + (\nu - \lambda)\Phi(\lambda,\nu)].$$

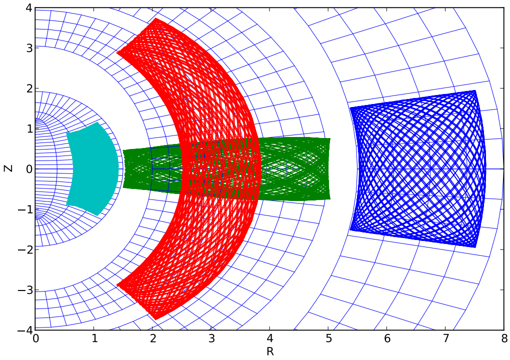
Under the approximation that the separation constant $I_3$ is indeed conserved along the orbit, actions are computed as

$$J_\lambda = \frac{1}{\pi}\int_{\lambda_{\rm min}}^{\lambda_{\rm max}} p_\lambda\,d\lambda, \quad J_\nu = \frac{1}{\pi}\int_{\nu_{\rm min}}^{\nu_{\rm max}} p_\nu\,d\nu.$$

[1] Note that the potential of the Perfect Ellipsoid [de Zeeuw 1985] is of the Stäckel form, but it is only one example of a much wider class of potentials.

# Stäckel fudge in practice

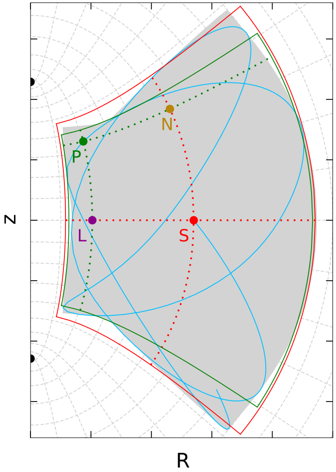
A rather flexible approximation: for each orbit, we have the freedom of using two functions $f_1(\lambda)$, $f_2(\nu)$ (directly evaluated from the actual potential $\Phi(R, z)$) to describe the motion in two independent directions. These functions are different for each orbit (implicitly depend on $E, L_z, I_3$). Moreover, we may choose the focal distance $\Delta$ of the auxiliary prolate spheroidal coordinate system for each orbit independently.

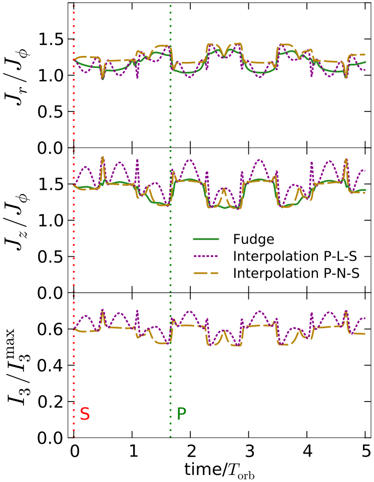


# Accuracy of the Stäckel fudge

Accuracy of action conservation using the Stäckel fudge: $\lesssim 1\%$ for most disk orbits, $\lesssim 10\%$ even for high-eccentricity orbits [except near resonances!].

Interpolation of $J_r, J_z$ on a 3d grid of $E, L_z, I_3$: 10x speed-up at the expense of a moderate [not always acceptable!] decrease in accuracy.



# Other methods for action computation

The accuracy of the Stäckel approximation is "uncontrollable" (cannot be systematically improved), and it is mainly used in axisymmetric potentials.

However, actions offer the only **systematic** method for computing the integrals of motion in a **non-perturbative** way for an arbitrary potential:

- Introduce a simple enough "toy" potential $\Phi^t$ (e.g., isochrone), for which the mapping between position–velocity $\{\mathbf{x}, \mathbf{v}\}$ and action–angle $\{\mathbf{J}^t, \boldsymbol{\theta}^t\}$ coordinates is known analytically.
- We seek a canonical transformation between the true (yet unknown) $\{\mathbf{J}, \boldsymbol{\theta}\}$ and the "toy" $\{\mathbf{J}^t, \boldsymbol{\theta}^t\}$. This transformation is described by a generating function $S(\mathbf{J}, \boldsymbol{\theta}^t)$, which can be expanded into Fourier series in $\boldsymbol{\theta}^t$:
  $S(\mathbf{J}, \boldsymbol{\theta}^t) = \mathbf{J} \cdot \boldsymbol{\theta}^t + \sum_{\mathbf{n}} S_{\mathbf{n}}(\mathbf{J}) \exp(i\mathbf{n} \cdot \boldsymbol{\theta}^t)$, where $\mathbf{n}$ are triplets of integers.
- Choose the Fourier coefficients $S_{\mathbf{n}}$ up to some maximum order $\mathbf{n}$ to approximate the true Hamiltonian to any desired accuracy.
- The transformation is given by $\mathbf{J}^t = \partial S / \partial \boldsymbol{\theta}^t$, $\boldsymbol{\theta} = \partial S / \partial \mathbf{J}$.

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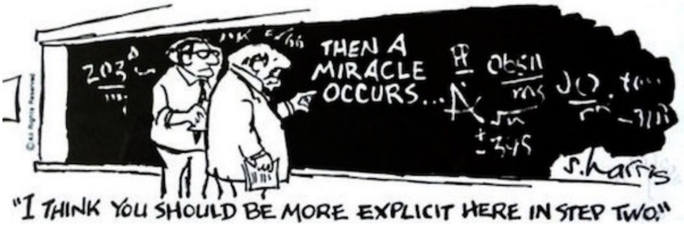



There are several variants of these methods, but we won't go into details.

- First numerically integrate the orbit, then obtain the coefficients $S_{\mathbf{n}}$ to minimize the variation of the toy Hamiltonian across the real orbit [Sanders & Binney 2014; Bovy 2014]. This gives the transformation from $\{\mathbf{x}, \mathbf{v}\}$ to $\{\mathbf{J}, \boldsymbol{\theta}\}$.
- Reverse transformation (torus mapping) allows one to compute the position/velocity from action/angle without the need to integrate an orbit [McGill & Binney 1990; McMillan & Binney 2008].
- A variation of this approach also works for resonantly-trapped orbits [Kaasalainen 1994; Binney 2016, 2018].

# Advantages of action/angle variables

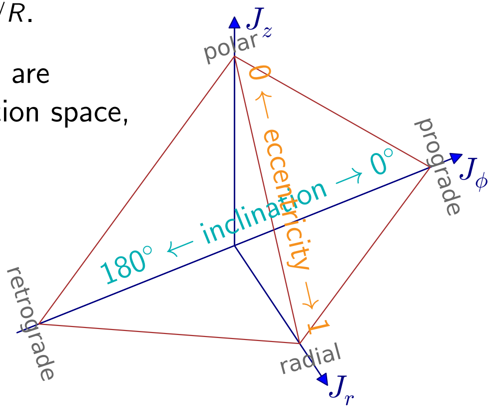
- Clear physical meaning (describe the extent of oscillations in each dimension).
- Most natural description of motion (angles change linearly with time).
- Possible range for each action variable is $[0..\infty)$ or $(-\infty..\infty)$, independently of the other ones (unlike $E$ and $L$, say).
- Canonical coordinates $\Rightarrow$ the 6d phase-space volume element is $d^3x\ d^3v = d^3J\ d^3\theta$.
- Actions are adiabatic invariants (are conserved under slow variation of potential).
- Perturbation theory most naturally formulated in terms of actions.
- Efficient methods for conversion between $\{\mathbf{x}, \mathbf{v}\}$ and $\{\mathbf{J}, \boldsymbol{\theta}\}$ now exist.

# Fun facts / rules of thumb about actions

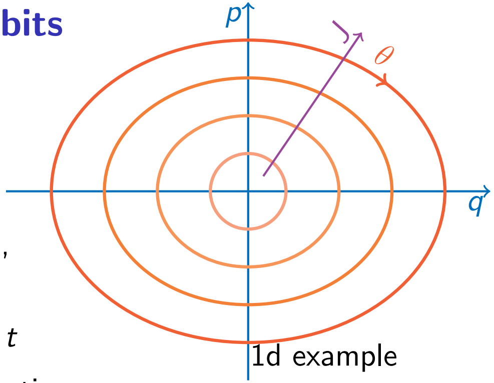
- Dimension of actions is length×velocity:
  if a star at a galactocentric distance $r$ travels with velocity $v$, then [at least one of the actions] $J \sim r\,v$.

- Frequencies: $\Omega_i(\mathbf{J}) = \partial H/\partial J_i$
  characteristic velocity $v_i \sim \sqrt{\Omega_i\,J_i}$
  e.g., for a circular orbit $J_\phi = R\,v_\phi$, $\Omega_\phi = v_\phi/R$.

- Surfaces of constant energy $H(\mathbf{J}) = E$ are approximately tetrahedra in the 3d action space, with $E \approx E(\Omega_r\,J_r + \Omega_z\,J_z + \Omega_\phi\,J_\phi)$.



# Natural coordinates to describe orbits



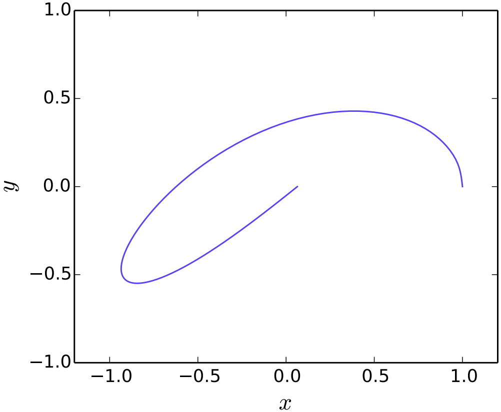

- The entire 6d phase space is foliated into non-intersecting 3d orbital tori.
- Actions tell you which orbit the star is on, angles – where it is located on this orbit.
- Angles change linearly with time, $\theta_i = \Omega_i\, t$
- Torus construction provides the transformation $\mathbf{J}, \boldsymbol{\theta} \to \mathbf{x}, \mathbf{v}$, i.e., one can find the position–velocity at any time without the need to integrate the orbit.
- In a time-averaged sense, only actions are significant (distribution of stars is averaged over angles – phase mixed);

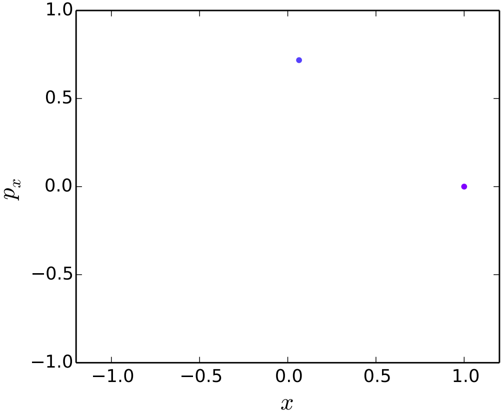
  however, for an initially localized ensemble of stars (e.g., a stream from a disrupted cluster), the distribution over angles is not uniform.

# Digression: the Poincaré surface of section

A convenient tool for analyzing orbits in 2d Hamiltonian systems at a fixed $E$ (e.g., motion in the equatorial plane, or in the meridional plane of an axisymmetric potential at a fixed $L_z$)

1. Numerically integrate the trajectory: $x(t), y(t), p_x(t), p_y(t)$.
2. Every time it passes through the axis $y = 0$ with $\dot{y} > 0$, put a point on the $x, p_x$ plane.



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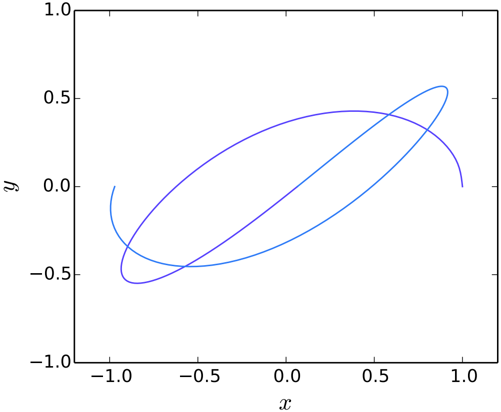


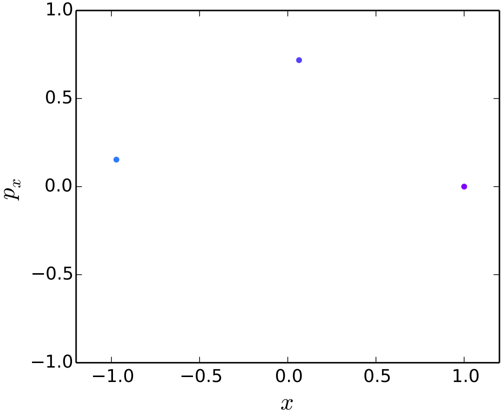

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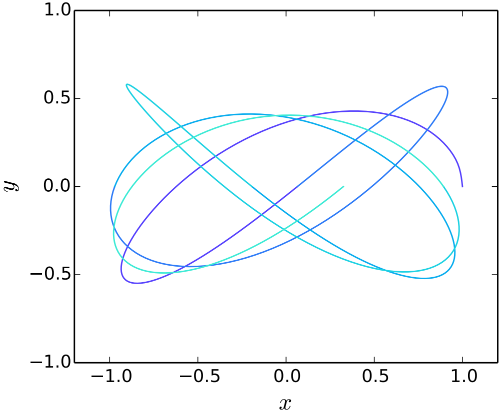


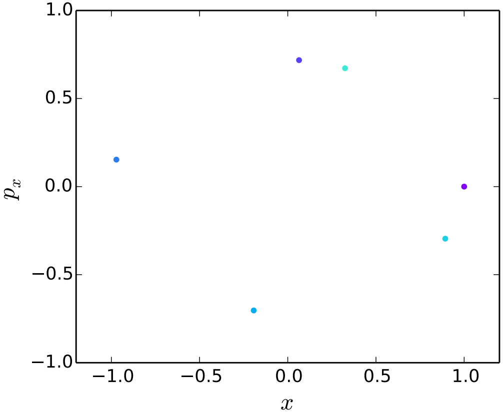

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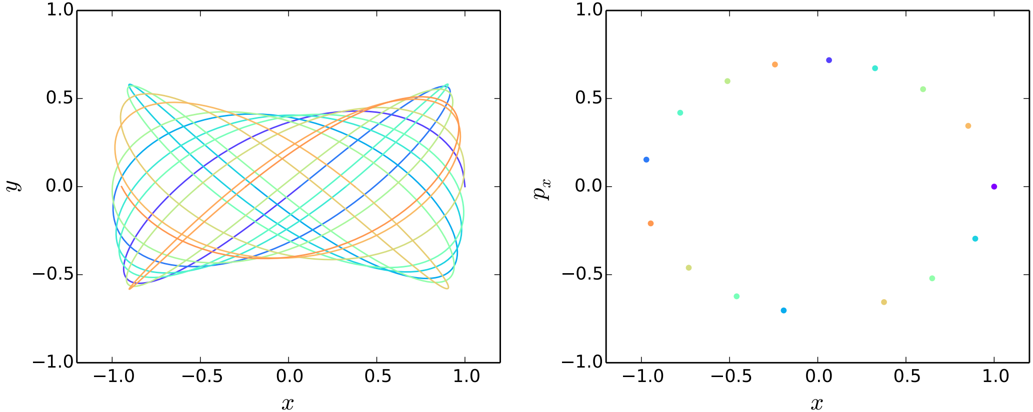

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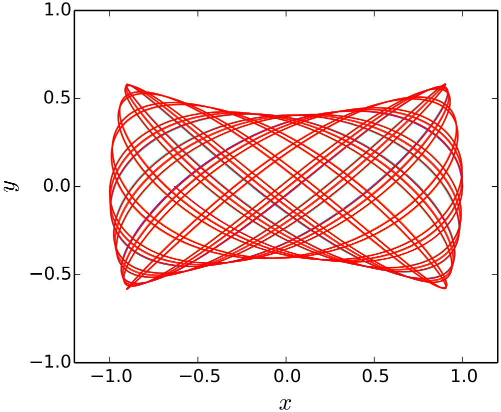
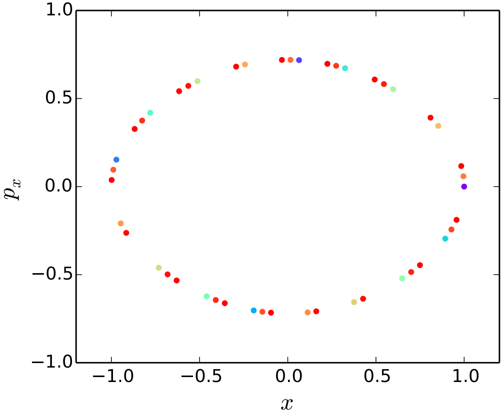


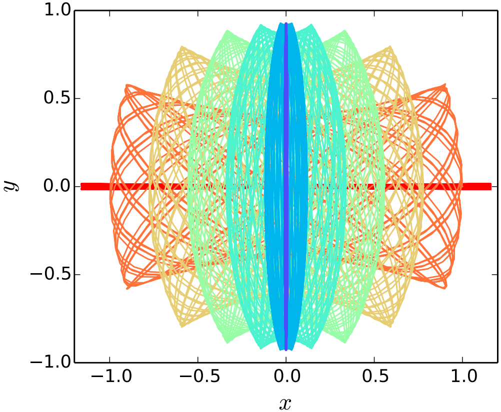



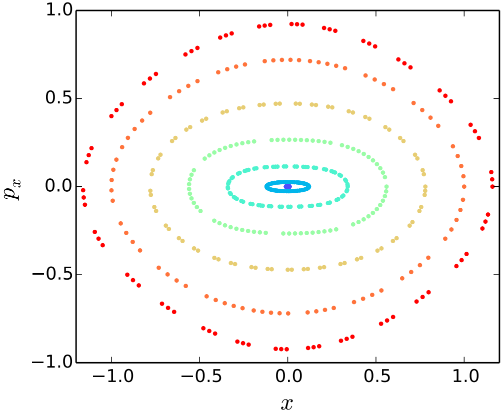
# Digression: the Poincaré surface of section

A convenient tool for analyzing orbits in 2d Hamiltonian systems at a fixed $E$ (e.g., motion in the equatorial plane, or in the meridional plane of an axisymmetric potential at a fixed $L_z$)

**3.** Each orbit corresponds to a closed loop in this plane.

**4.** Repeat for many different initial conditions to get the "phase portrait" of the Hamiltonian.



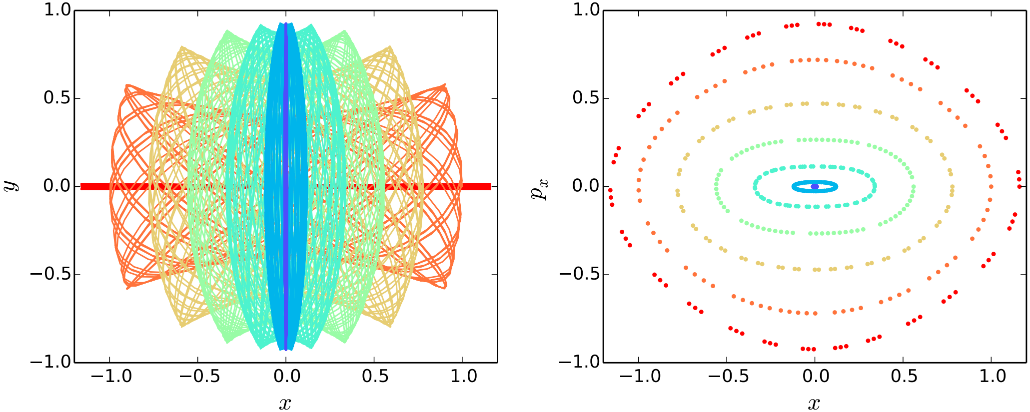



# Digression: the Poincaré surface of section

A convenient tool for analyzing orbits in 2d Hamiltonian systems at a fixed $E$

(e.g., motion in the equatorial plane, or in the meridional plane of an axisymmetric potential at a fixed $L_z$)

3. Each orbit corresponds to a closed loop in this plane.
5. The action of an orbit is just the area inside its Poincaré curve:
   $J_x = \frac{1}{2\pi} \oint p_x \, dx$



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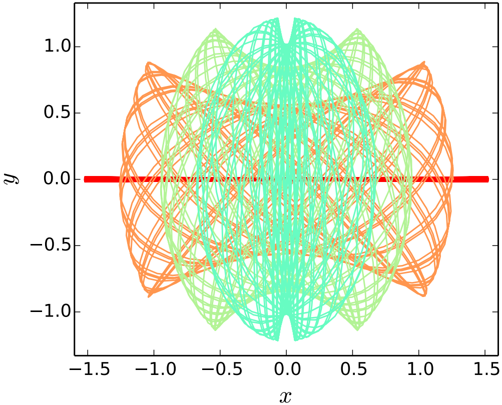
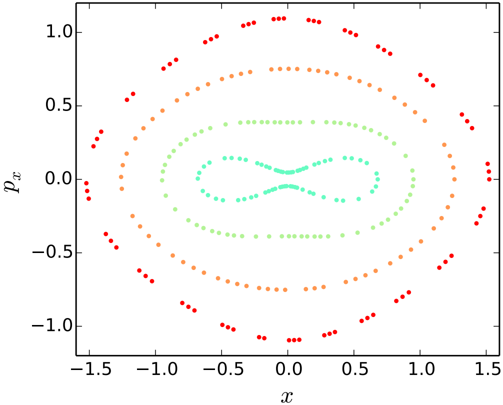
**6.** Now repeat this exercise for a different choice of energy $E$



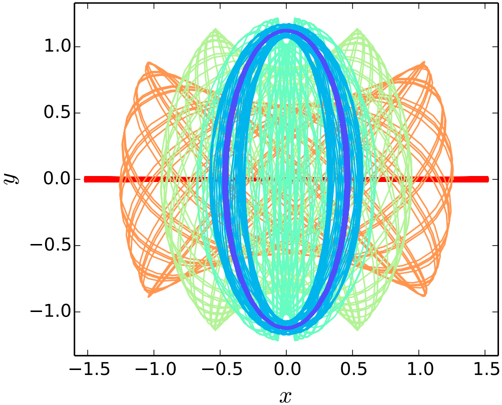



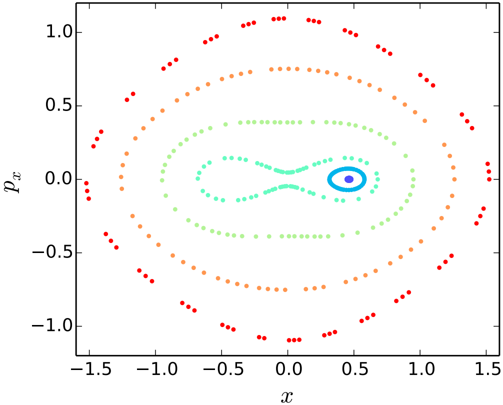
# Digression: the Poincaré surface of section

A convenient tool for analyzing orbits in 2d Hamiltonian systems at a fixed $E$

(e.g., motion in the equatorial plane, or in the meridional plane of an axisymmetric potential at a fixed $L_z$)

7. This portrait may contain more than one orbit family!
8. The meaning of actions is different for each orbit family (e.g., it is $J_x$ for a box orbit, and $J_r$ for a loop orbit)



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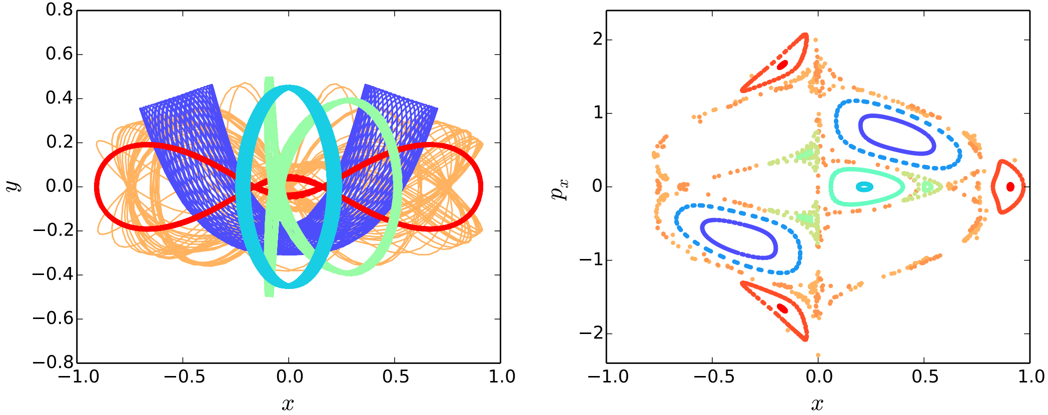
(e.g., motion in the equatorial plane, or in the meridional plane of an axisymmetric potential at a fixed $L_z$)

7. This portrait may contain more than one orbit family!
9. For high-order resonances the action *may* describe the width around parent orbit, and for chaotic orbits the actions are not defined at all..



# Action–angles are canonical coordinates

A distribution function (DF) in phase space $f(\mathbf{x}, \mathbf{v})$ can be expressed in any other coordinates, e.g., $f(E, L)$ or $f(\mathbf{J}, \theta)$.

In a phase-mixed system, it is independent of $\boldsymbol{\theta}$.

The mass in a given volume of phase space is $\int f(\mathbf{x}, \mathbf{v})\, d^3x\, d^3v = \int f(\mathbf{J}, \theta)\, d^3J\, d^3\theta = (2\pi)^3 \int f(\mathbf{J})\, d^3J$ –
does not depend on the potential and has no extra multiplicative factors!
(e.g., compared to $\int f(E, L)\, 4\pi^2\, T(E, L;\, \Phi)\, dL^2$ for a conventional DF).

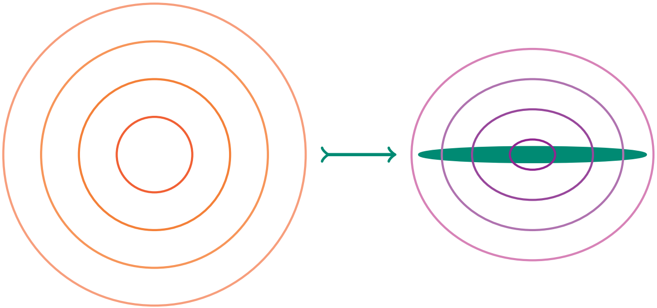
The functional form of the DF is independent of the potential $\Phi$, but the correspondence between $\mathbf{x}, \mathbf{v}$ and $\mathbf{J}, \boldsymbol{\theta}$, of course, does depend on $\Phi$.

# Adiabatic invariance of actions

Actions are conserved under slow variations of potential ($\Omega\,\tau \gg 1$).

Examples:

- accreted satellite galaxies should stay localized in the action space;
- compression of the dark halo after the formation of the stellar disk:



# Perturbation theory in action space

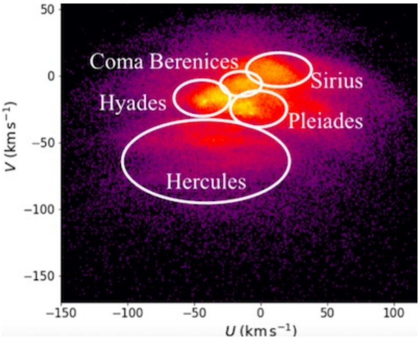
$$f(\mathbf{J},\boldsymbol{\theta},t) = f_0(\mathbf{J}) + \epsilon f_1(\mathbf{J},\boldsymbol{\theta},t),$$
$$H(\mathbf{J},\boldsymbol{\theta},t) = H_0(\mathbf{J}) + \epsilon H_1(\mathbf{J},\boldsymbol{\theta},t) = H(\mathbf{x},\mathbf{v},t) \equiv \Phi_0(\mathbf{x}) + \epsilon\Phi_1(\mathbf{x},t) + \tfrac{1}{2}v^2.$$

Linearized Vlasov / collisionless Boltzmann equation:

$$0 = \frac{\partial f}{\partial t} + [H, f] \approx \frac{\partial f_1}{\partial t} + \frac{\partial f_1}{\partial \boldsymbol{\theta}}\frac{\partial H_0}{\partial \mathbf{J}} - \frac{\partial f_0}{\partial \mathbf{J}}\frac{\partial \Phi_1}{\partial \boldsymbol{\theta}}.$$

$\Phi_1(\mathbf{x},t)$ is the external perturbation augmented with internal self-gravity (diverges at resonances!).

For the given $f_0$ and $\Phi_1$, one may compute the perturbed DF $f_1(\mathbf{J},\boldsymbol{\theta},t)$ [e.g., Monari+ 2016–2018] so far has only been done under epicyclic approximation, but a Stäckel generalization is possible.



# Distribution functions

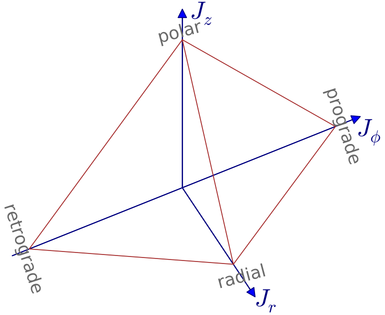
The pair $f(\mathbf{J})$, $\Phi(\mathbf{x})$ provides the complete description of the system:

- $\Phi$ determines the transformation $\{\mathbf{x},\mathbf{v}\} \leftrightarrow \{\mathbf{J},\boldsymbol{\theta}\}$;
- density is $\rho(\mathbf{x}) = \int f\left[\mathbf{J}(\mathbf{x},\mathbf{v};\ \Phi)\right]\ d^3v$;
- velocity moments are $\overline{\mathbf{v}} = \dfrac{1}{\rho}\displaystyle\int f\left[\mathbf{J}(\mathbf{x},\mathbf{v};\ \Phi)\right]\ \mathbf{v}\ d^3v$;
  $\overline{v_i\, v_j} = \dfrac{1}{\rho}\displaystyle\int f\left[\mathbf{J}(\mathbf{x},\mathbf{v};\ \Phi)\right]\ v_i\, v_j\ d^3v$, etc.

Now two questions remain:

1. How to choose a sensible $f(\mathbf{J})$
2. How to find $\Phi(\mathbf{x})$ consistent with this DF

# Distribution functions for spheroidal systems




Recall that surfaces of constant energy are approximately tetrahedra in action space, $E \approx E(\Omega_r J_r + \Omega_z J_z + \Omega_\phi J_\phi)$.

So if we consider $f(\mathbf{J}) = f_0\big[h(\mathbf{J})\big]$, where $h(\mathbf{J}) = k_r J_r + k_z J_z + k_\phi |J_\phi|$ is a linear combination of three actions with the above coefficients, it will be approximately isotropic (dependent on energy only).

We may construct tailored anisotropic systems by changing the coefficients $k_i$. [Binney 2014, Posti+ 2015, Williams & Evans 2015].

The function $f_0$ is responsible for the overall density profile of the system; a reasonable choice is a double-power-law model:

$$f_0(h) \propto \frac{h^\Gamma}{\big(1 + [h/h_0]^\eta\big)^{(\Gamma - \mathrm{B})/\eta}}$$

$\Gamma$: inner slope
B: outer slope
$h_0$: scale action

# Distribution functions for disky systems

In the epicyclic approximation, the motion is separable in $R, z$, and the DF in each dimension has a nearly Boltzmann form:
$f(E, L_z, E_z) \propto f_0(L_z)\ \exp(-E_R/\sigma_R^2)\ \exp(-E_z/\sigma_z^2)$.

One may construct a similarly behaving DF expressed in terms of actions, replacing $E_R \to \Omega_R\, J_R,\ E_z \to \Omega_z\, J_z$ – a quasi-isothermal DF [Binney & McMillan 2011]:

$$f(\mathbf{J}) = \frac{\tilde\Sigma\,\Omega}{2\pi^2\,\kappa^2} \times \frac{\kappa}{\tilde\sigma_r^2} \exp\left(-\frac{\kappa\, J_r}{\tilde\sigma_r^2}\right) \times \frac{\nu}{\tilde\sigma_z^2} \exp\left(-\frac{\nu\, J_z}{\tilde\sigma_z^2}\right) \times \begin{cases} 1 & \text{if } J_\phi \ge 0, \\ \exp\left(\frac{2\Omega\, J_\phi}{\tilde\sigma_r^2}\right) & \text{if } J_\phi < 0, \end{cases}$$

$$\tilde\Sigma(R_c) \equiv \Sigma_0 \exp\left(-\frac{R_c}{R_\mathrm{disk}}\right), \quad \tilde\sigma_r^2(R_c) \equiv \sigma_{r,0}^2 \exp\left(-\frac{2R_c}{R_{\sigma,r}}\right), \quad \tilde\sigma_z^2(R_c) \equiv 2\, h_\mathrm{disk}^2\, \nu^2(R_c).$$

It produces nearly exponential radial profiles of surface density and nearly isothermal vertical density profiles: $\rho(R, z) \propto \exp\left(-\dfrac{R}{R_\mathrm{disk}}\right)\ \mathrm{sech}^2\left(\dfrac{z}{h}\right)$.

# Self-consistent models

distribution function

integals of motion

gravitational potential

**1.** Collisionless Boltzmann equation:

$$\mathbf{v}\,\frac{\partial f}{\partial \mathbf{x}} - \frac{\partial \Phi}{\partial \mathbf{x}}\,\frac{\partial f}{\partial \mathbf{v}} = 0 \quad \Longrightarrow \quad f = f\big(\mathcal{I}(\mathbf{x},\mathbf{v};\ \Phi)\big).$$

(Assumption: a galaxy is a collisionless system in a steady state)

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# Self-consistent models – iterative approach

1. Assume a particular distribution function $f\left(\mathcal{I}\right)$;
2. Adopt an initial guess for $\Phi(\mathbf{x})$;
3. Establish the integrals of motion $\mathcal{I}(\mathbf{x},\mathbf{v})$ in this potential;
4. Compute the density $\rho(\mathbf{x}) = \iiint d^3v\ f\left(\mathcal{I}(\mathbf{x},\mathbf{v})\right)$;
5. Solve the Poisson equation to find the new potential $\Phi(\mathbf{x})$;
6. Repeat until convergence.

Origin: Prendergast & Tomer 1970;

used in Kuijken & Dubinski 1995, Widrow+ 2008, Taranu+ 2017 (GalactICs),

Piffl+ 2014, Cole & Binney 2016, Sanders & Evans 2016 (action-based formalism).

# Advantages of models based on distribution function

- Clear physical meaning
  (localized structures in the space of integrals of motion);
- Easy to compare different models
  (how to compare two $N$-body or $N$-orbit models?);
- Easy to compare models to discrete observational data;
- Easy to sample particles from the distribution function
  (convert to an $N$-body model);
- Stability analysis
  (perturbation theory most naturally formulated in terms of actions);

# Caveats:

- Implicitly rely on the integrability of the potential,
  ignore the presence of resonant orbit families (but see Binney 2016, 2018);
- So far implemented only for axisymmetric models
  (not a fundamental limitation).

# Galactic modelling tasks

| | | | | | |
|---|---|---|---|---|---|
| ✓ | ✓ | ✓ | ✓ | ✓ | ▶ Gravitational potentials and forces |
| ✓ | ✓ | ✓ | ✓ | ✓ | ▶ Orbit integration and analysis |
| ✓ | ✓ | ✓ | ✓ | ✓ | ▶ Conversion between position–velocity and action–angle variables |
| ✓ | | ✓ | | ✓ | ▶ Distribution functions |
| | | ✓ | ✓ | | ▶ Streams modelling |
| | | | | ✓ | ▶ DF-based self-consistent models |
| | | | | ✓ | ▶ Orbit-superposition (Schwarzschild) models |
| | | ✓ | | ✓ | ▶ Jeans models |

# Galactic modelling software

Torus Mapper [McMillan & Binney 2008; Binney & McMillan 2016]

TACT (the Action Computation Toolbox) [Sanders & Binney 2012–2016]

Galpy [Bovy 2015]

Gala [Price-Whelan 2017]

Agama [Vasiliev 2019]

# AGAMA – All-purpose galaxy modeling architecture

- Extensive collection of gravitational potential models
  (analytic profiles, azimuthal- and spherical-harmonic expansions)
  constructed from smooth density profiles or $N$-body snapshots;
- Conversion to/from action/angle variables;
- Self-consistent multicomponent models with action-based DFs;
- Schwarzschild orbit-superposition models;
- Generation of initial conditions for $N$-body simulations;
- Various math tools: 1d,2d,3d spline interpolation, penalized spline fitting and density estimation, multidimensional sampling;
- Efficient and carefully designed C++ implementation, examples, Python and Fortran interfaces, plugins for Galpy, NEMO, AMUSE.

`arXiv:1802.08239`, `1802.08255`

`https://github.com/GalacticDynamics-Oxford/Agama`