TIMER collaboration meeting, Madrid, August 2026

Dynamical modelling with distribution functions and orbits

Eugene Vasiliev

University of Surrey

Joan Miró – Constellations

Chapter 1: Distribution function-based models

A stellar system composed of a large number $N \gg 1$ of "identical" stars can be described by a distribution function (DF):

$f(\mathbf{x}, \mathbf{v};\ t)$ – probability density in the phase space (at time t),

$\int f(\mathbf{x}, \mathbf{v})\, \mathrm{d}^3x\, \mathrm{d}^3v = 1$ (unit-normalised) or M_total (mass-normalised).

For a multicomponent system, define a separate DF f_k for each species k, or an "extended distribution function" (EDF) with additional arguments:

$f(\mathbf{x}, \mathbf{v};\ \boldsymbol{\eta};\ t)$ ($\boldsymbol{\eta}$ may be age, mass, chemical composition, etc.)

DF offers a complete description of the stellar population:

density $\rho(\mathbf{x}) = \displaystyle\int f(\mathbf{x}, \mathbf{v})\, \mathrm{d}^3v$,

mean velocity $\overline{\mathbf{v}}(\mathbf{x}) = \dfrac{1}{\rho(\mathbf{x})} \displaystyle\int \mathbf{v}\, f(\mathbf{x}, \mathbf{v})\, \mathrm{d}^3v$,

second moment of velocity $\overline{v^2_{ij}}(\mathbf{x}) = \dfrac{1}{\rho(\mathbf{x})} \displaystyle\int v_i\, v_j\, f(\mathbf{x}, \mathbf{v})\, \mathrm{d}^3v$, etc.

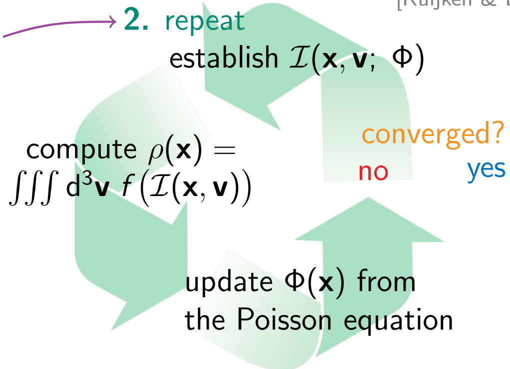
Constructing DF-based equilibrium models: two approaches

a. From the given *total* potential $\Phi(\mathbf{x})$ and density of *stellar tracers* $\rho(\mathbf{x})$, obtain $f\big(\mathcal{I}(\mathbf{x},\mathbf{v};\ \Phi)\big)$ (e.g. by [anisotropic] Eddington inversion).

b. From an assumed $f\big(\mathcal{I}(\mathbf{x},\mathbf{v};\ \Phi)\big)$,
compute the density $\rho(\mathbf{x}) = \iiint f\big(\mathcal{I}(\mathbf{x},\mathbf{v};\ \Phi)\big)\,\mathrm{d}^3\mathbf{v}$
and then the corresponding potential $\Phi(\mathbf{x})$ from the Poisson equation.

Some spherical isotropic models exist (polytropes, King models, etc.),
but in general this circular dependency is resolved using an iterative approach
[Kuijken & Dubinski 1995, Binney 2014].

1. assume $f(\mathcal{I})$ and an initial guess for Φ → **2.** repeat



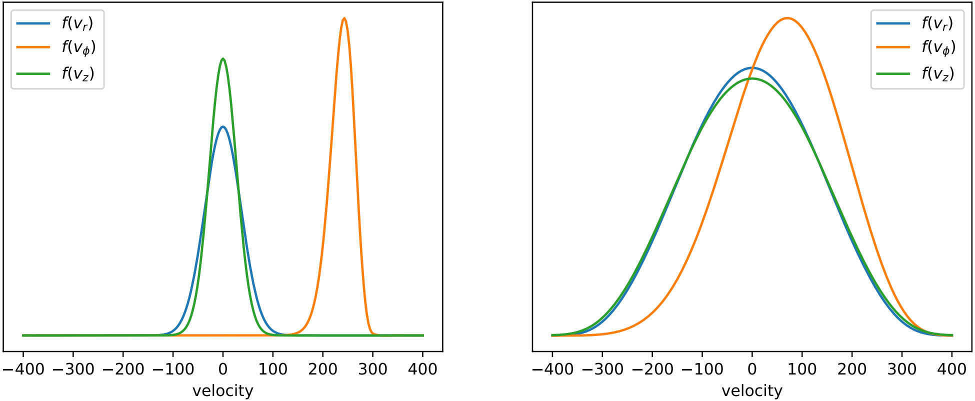
3. enjoy!

DFs for disky and spheroidal components

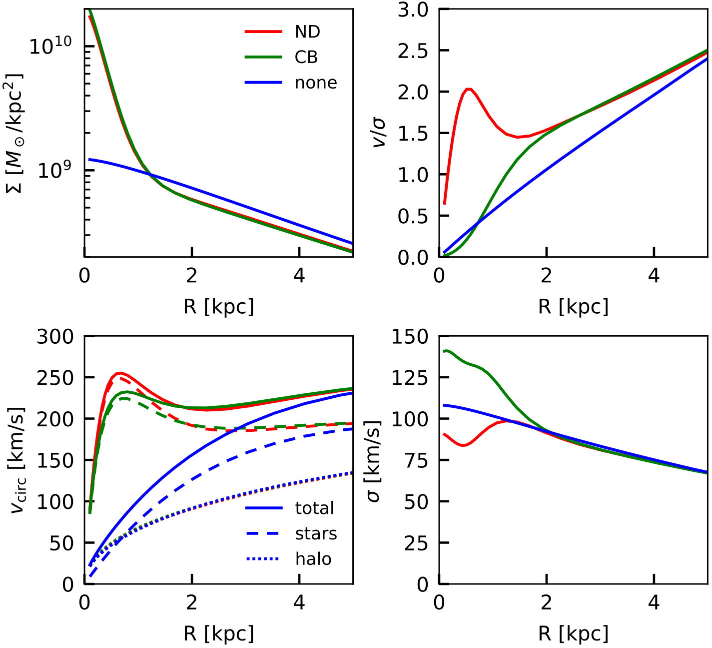
similar density profiles (except flattening), but very different kinematics:

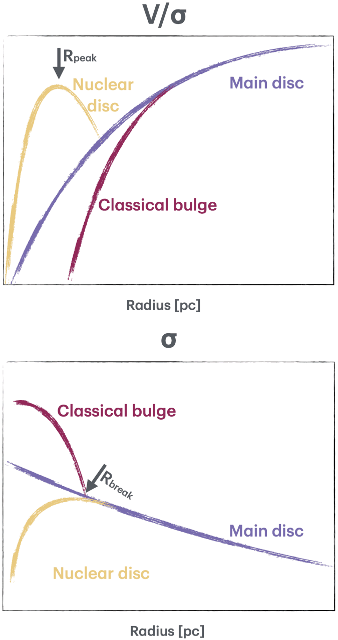
discs: $\overline{v_\phi}$ is much closer to $v_{\rm circ}$, and $\sigma_{r,\phi} \ll \overline{v_\phi}$ (except small radii);

spheroids ("classical bulges"): $\sigma_{r,\phi} \gg \overline{v_\phi}$ everywhere.



Observable kinematic signatures





[de Sá-Freitas+, in prep.]

Fitting DF-based models to observations

- Assume suitable parametric DF forms for all stellar components $f^{(c)}(\mathcal{I};\ \boldsymbol{\alpha})$, plus potentials $\Phi^{(d)}(\mathbf{x};\ \boldsymbol{\beta})$ of unseen components (SMBH, DM halo...).
- For each choice of DF and potential parameters $\boldsymbol{\alpha}, \boldsymbol{\beta}$:
  - build the total potential $\Phi = \sum \Phi^{(c)} + \sum \Phi^{(d)}$ iteratively from the DFs $f^{(c)}$;
  - construct the model kinematic *and* surface density maps;
  - compare with observations and compute $\chi^2(\boldsymbol{\alpha}, \boldsymbol{\beta})$.
- explore the parameter space $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\}$ using MCMC (or something similar).

Note that there is no separate photometric fitting step:
the DFs determine both kinematic and spatial structure.

DF-based models are currently limited to axisymmetry $\Longrightarrow$ no bars

Chapter 2: Schwarzschild orbit-superposition method

Introduced by Schwarzschild (1979) as a practical approach
for constructing self-consistent triaxial models with prescribed $\rho(\mathbf{x}) \Leftrightarrow \Phi(\mathbf{x})$.

integrals of motion

To invert the equation $\rho(\mathbf{x}) = \iiint f\left(\mathcal{I}\left(\mathbf{x}, \mathbf{v}\,;\ \Phi\right)\right)\, d^3\mathbf{v}$,

discretize both the density profile and the distribution function:

$\rho(\mathbf{x}) \implies$ cells of a spatial grid;
mass of each cell is $M_c = \iiint\limits_{\mathbf{x}\in V_c} \rho(\mathbf{x})\, d^3x$;

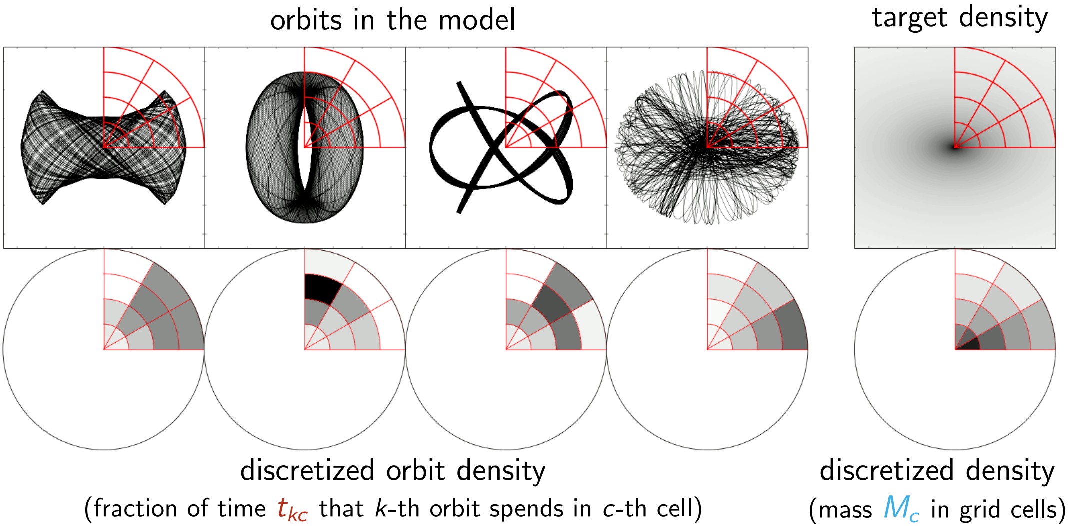
$f(\mathcal{I}) \implies$ collection of orbits with unknown weights:

$$f(\mathcal{I}) = \sum_{k=1}^{N_{\rm orb}} w_k\ \delta(\mathcal{I} - \mathcal{I}_k)$$

each orbit is a delta-function in the space of integrals of motion

adjustable weight of each orbit [to be determined]

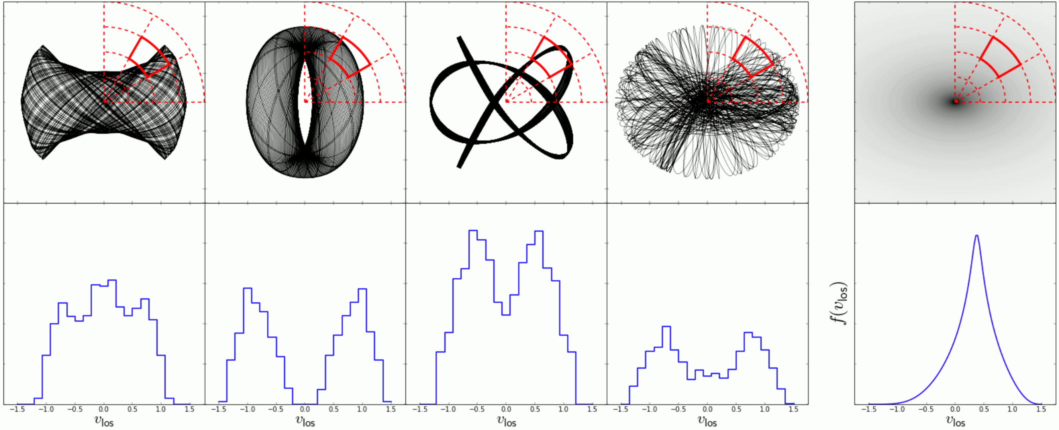
Schwarzschild's orbit-superposition method: self-consistency



For each c-th cell we require $\sum_k w_k\, t_{kc} = M_c$, where $w_k \geq 0$ is orbit weight

Schwarzschild's orbit-superposition method: kinematics

orbits in the model

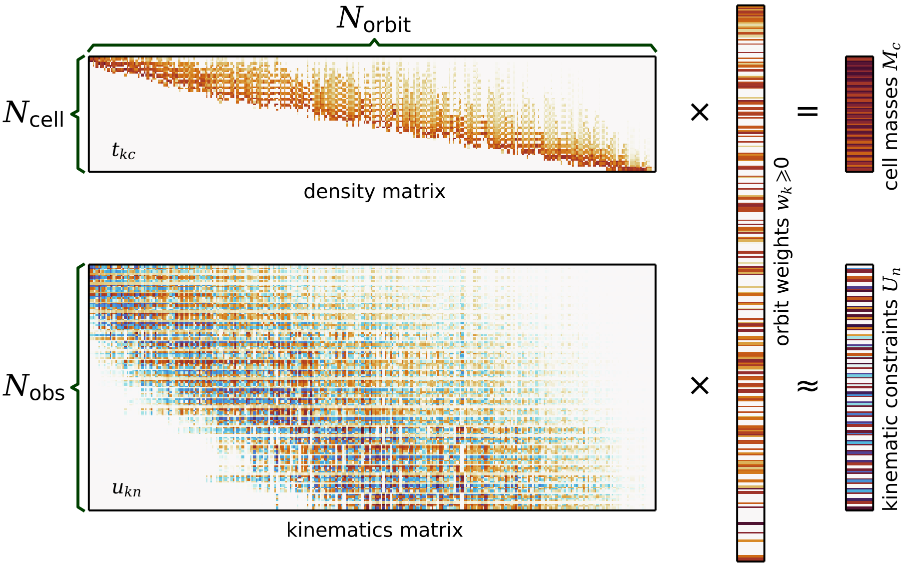


orbit LOSVDs

target LOSVD

Schwarzschild's orbit-superposition method: fitting procedure

Solve the linear system with non-negativity constraints on the solution vector $w_k \geq 0$ (linear or non-linear optimization problem)



Schwarzschild's orbit-superposition method: fitting procedure

- Assume some potential $\Phi(\mathbf{x})$
  (e.g., from the deprojected luminosity profile plus parametric DM halo or SMBH)
- Construct the orbit library in this potential:
  for each k-th orbit, store its contribution to the discretized density profile
  t_{kc}, $c = 1..N_{\rm cell}$ and to the kinematic observables u_{kn}, $n = 1..N_{\rm obs}$
- Solve the constrained optimization problem to find orbit weights w_k:

$$\text{minimize } \chi^2 + \mathcal{S} \equiv \sum_{n=1}^{N_{\rm obs}} \left(\frac{\sum_{k=1}^{N_{\rm orb}} w_k\, u_{kn} - U_n}{\delta U_n} \right)^2 + \mathcal{S}(\{w_k\})$$

regularization term

observational constraints

their uncertainties

$$\text{subject to } w_k \ge 0, \quad k = 1..N_{\rm orb},$$

$$\sum_{k=1}^{N_{\rm orb}} w_k\, t_{kc} = M_c, \quad c = 1..N_{\rm cell}$$

density constraints (cell masses)

- Explore the parameter space (M/L, $M_{\rm SMBH}$, i, ...) around minimum χ^2

Schwarzschild's orbit-superposition method: implementations

- spherical codes: Richstone & Tremaine 1984; Rix+ 1997; Breddels & Helmi 2013; ...
- axisymmetric: "Leiden" [van der Marel, Cretton, Cappellari, ... – since 1998]
- axisymmetric: "Nukers" [Gebhardt, Richstone, Kormendy, ... – since 2000]
- axisymmetric: "MasMod" [Valluri, Merritt, Emsellem – since 2004]
- triaxial/Milky Way bar: Zhao, Wang, Mao 1996, 2012
- triaxial: van den Bosch, van de Ven, de Zeeuw, Zhu, ... – since 2008 $\Longrightarrow$
  - "Dynamite" [Jethwa+ 2020; Thater+ 2022; Tahmasebzadeh+ 2022]
  - "TriOS" [Quenneville+ 2022]
- triaxial: "SMART" [Lipka & Thomas 2021; Neureiter+ 2021; Tikhonenko+ 2026]
- triaxial: "Forstand" [Vasiliev & Valluri 2020]

For an in-depth introduction...

github.com/GalacticDynamics-Oxford/Agama/blob/master/py/tutorial_schwarzschild.ipynb

master | Agama / py / tutorial_schwarzschild.ipynb | Go to file

Preview | Code | Blame | 3364 lines (3364 loc) · 929 KB | Raw

Schwarzschild orbit-superposition modelling

This approach for constructing equilibrium models was introduced by Martin Schwarzschild in 1979 (after his retirement!), as the first practical demonstration that triaxial equilibrium galaxy models may exist – at the time, it was not at all obvious!

Originally, the method was purely a theorist's tool, but soon it was extended to take into account observational constraints and became one of standard tools in dynamical modelling (inferring the mass distribution from kinematic data). At the same time, its theoretical applications include the construction of initial conditions for N-body simulations. In both theoretical and observational flavours, its advantages are the ability to handle complicated geometry of the model (axisymmetric, triaxial and even a rotating bar) and a very flexible representation of the distribution function – the latter can also be a drawback, in the sense that the resulting models may have unphysically rapid variations of the DF across the phase space. To mitigate this situation, some regularisation is needed, as with any non-parametric method.

In this tutorial, we first discuss the fundamental concepts of the orbit-superposition method and the particular features of its implementation in `Agama` – essentially, the "theoretical" flavour of the method. Observational applications will be considered in the second part of this tutorial. Since the method works with orbits, it is essential to first complete the `tutorial_potential_orbits` notebook.

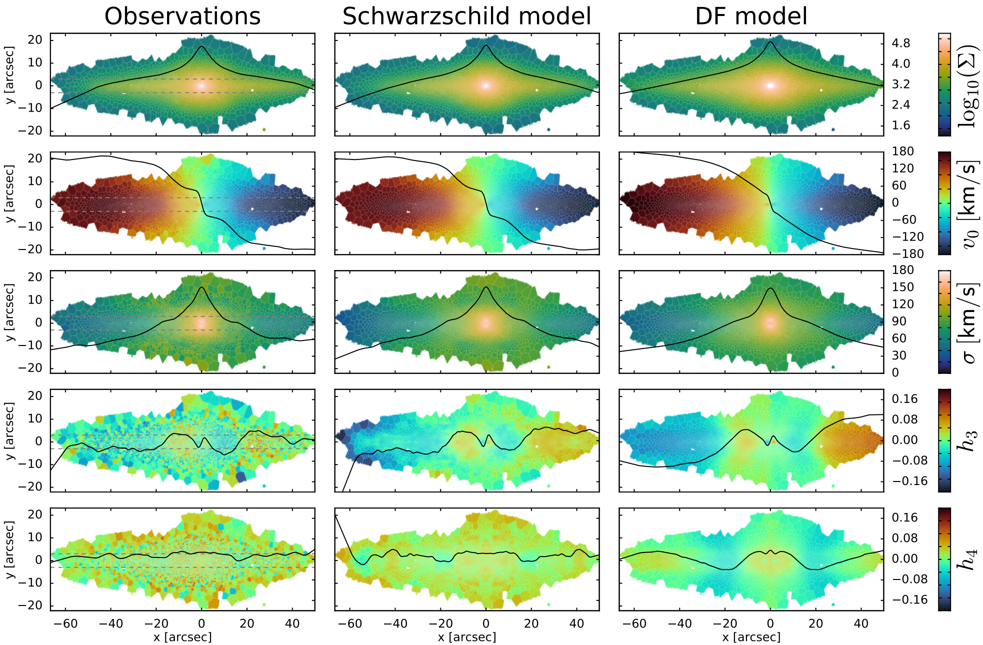
(sufficiently general in scope, but with a focus on the FORSTAND code)

`https://eugvas.net/docs/tutorial_schwarzschild_part1.mp4, part2.mp4`

Comparison of Schwarzschild and DF-based models

arbitrary combination of orbits	prescribed analytic form for the DFs
need post-processing (e.g. radius vs. circularity plots) to understand the structure of the model	interpretable parameters of the DF (e.g. velocity dispersions, anisotropy)
stellar density is usually fitted beforehand and is only scaled by M/L	stellar density and potential are determined by the DF, observed density is fitted simultaneously with kinematics
free parameters are only for the unseen mass components and stellar M/L	larger parameter space (5–10 for each DF component)
can be triaxial and barred	currently limited to axisymmetry

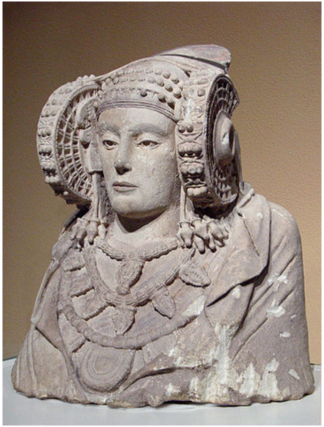
Comparison of Schwarzschild and DF-based models



FCC 170 [Galán-de Anta+ 2023]

Modelling bars: The deprojection problem

$\Longrightarrow$

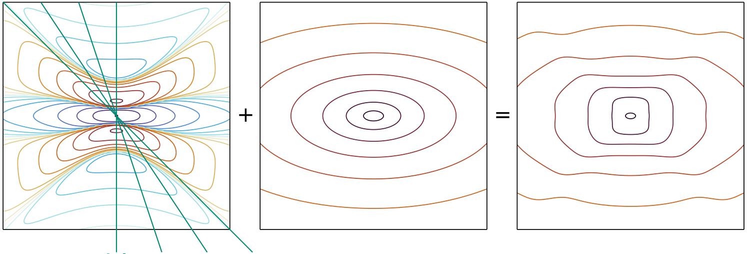


Pathway from 2d surface brightness profile to 3d density profile is non-unique

Approaches to deprojection

1. Ellipsoidal assumption:
   - first fit the surface density Σ by a combination of ellipsoidally stratified components of the form $\Sigma(R,\phi) = f(\tilde{R})$, where $f(\tilde{R})$ is some function (e.g. Gaussian), $\tilde{R}^2 \equiv R^2\big[\cos^2(\phi-\phi_0) + (1-e^2)^{-1}\sin^2(\phi-\phi_0)\big]$, e is [apparent] ellipticity, and ϕ_0 is the orientation of the major axis.
   - then for each choice of assumed orientation angles, each component can be *uniquely* deprojected into a 3d ellipsoidal profile $\rho\big(\sqrt{x^2+(y/p)^2+(z/q)^2}\big)$.

The often-forgotten problem is that this deprojection is unique *only* within the class of ellipsoidally stratified functions



konus densities: invisible at some orientations [Kochanek & Rybizki 1996; Gerhard & Binney 1996]

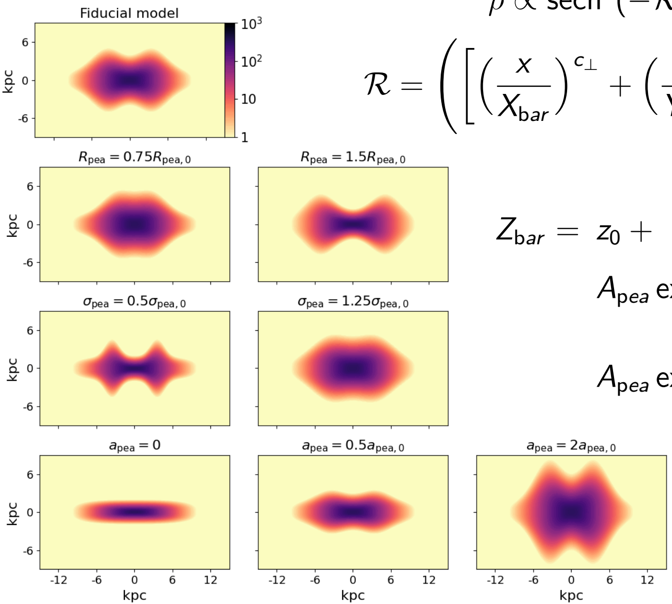
Approaches to deprojection

1. Ellipsoidal assumption:
   - first fit the surface density Σ by a combination of ellipsoidally stratified components of the form $\Sigma(R,\phi) = f(\tilde{R})$, where $f(\tilde{R})$ is some function (e.g. Gaussian), $\tilde{R}^2 \equiv R^2\big[\cos^2(\phi-\phi_0) + (1-e^2)^{-1}\sin^2(\phi-\phi_0)\big]$, e is [apparent] ellipticity, and ϕ_0 is the orientation of the major axis.
   - then for each choice of assumed orientation angles, each component can be *uniquely* deprojected into a 3d ellipsoidal profile $\rho\big(\sqrt{x^2+(y/p)^2+(z/q)^2}\big)$.

2. Forward modelling:
   - choose a suitable functional form for $\rho(x,y,z;\ \mathbf{p})$;
   - assume some viewing angles $\boldsymbol{\psi}$ and parameters $\mathbf{p}$;
   - compute the projected profile $\Sigma(X,Y)$ and compare with observations;
   - explore the parameter space of $\boldsymbol{\psi}$ and $\mathbf{p}$ around the minimum χ^2.

Need a photometric fitting code that allows for custom components and can compute projection integrals (e.g. IMFIT, Erwin 2015) and a lot of patience...

X-shaped bar model [Robin+ 2012; Fragkoudi+ 2015]



$$\rho \propto \mathrm{sech}^2(-\mathcal{R})$$

$$\mathcal{R} = \left(\left[\left(\frac{x}{X_{\rm bar}} \right)^{c_\perp} + \left(\frac{y}{Y_{\rm bar}} \right)^{c_\perp} \right]^{c_\parallel / c_\perp} + \left(\frac{z}{Z_{\rm bar}} \right)^{c_\parallel} \right)^{1/c_\parallel}$$

$c_\parallel$, $c_\perp$: boxiness coefficients

$$Z_{\rm bar} = z_0 + A_{\rm pea} \exp\left(- \frac{(x - R_{\rm pea})^2 + y^2}{2 w_{\rm pea}^2} \right) + A_{\rm pea} \exp\left(- \frac{(x + R_{\rm pea})^2 + y^2}{2 w_{\rm pea}^2} \right)$$

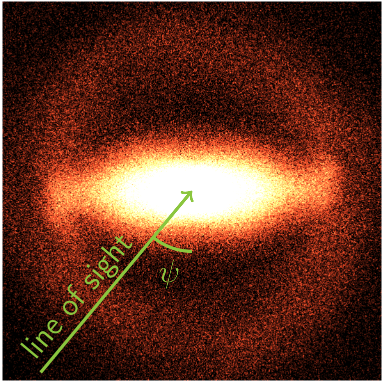
$A_{\rm pea}$: X/peanut amplitude

$R_{\rm pea}$: peanut location

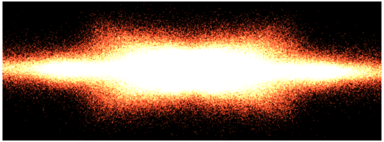
$w_{\rm pea}$: peanut width

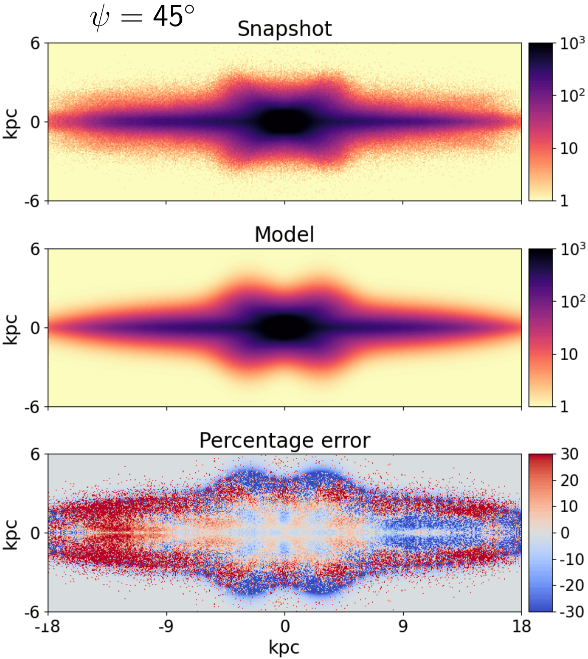
Forward-modelling of 3d density profile

face-on view



edge-on view, $\psi = 0$



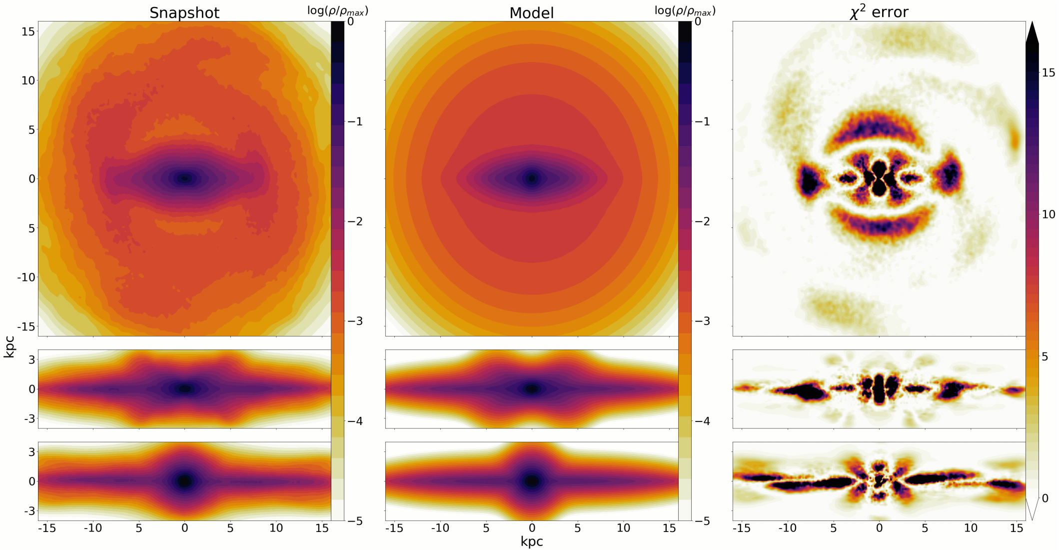


[Dattathri+ 2024]

Forward-modelling of 3d density profile

The fitted model qualitatively recovers the 3d density profile, though not without some defects

$\psi = 45°$



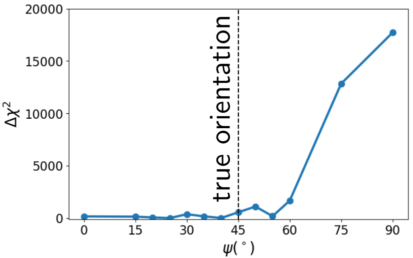
[Dattathri+ 2024]

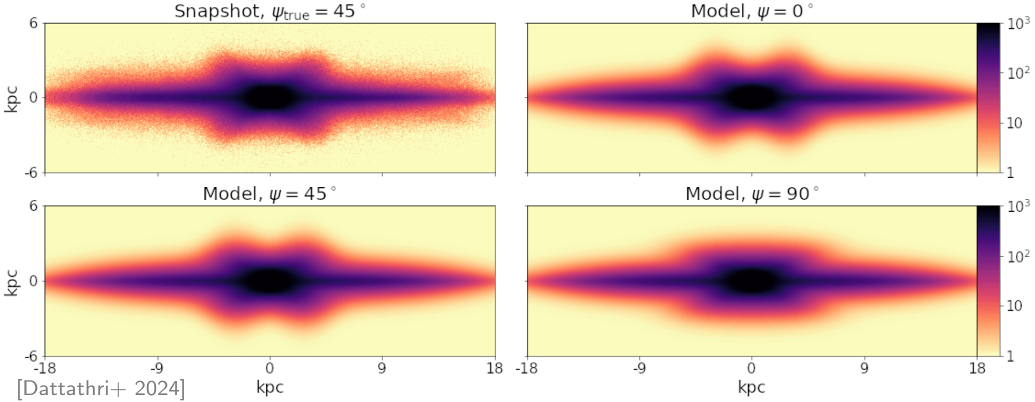
Degeneracies in determining bar orientation

It is impossible to distinguish a rotated bar ($0 < \psi < i_{max} \lesssim 90^\circ$) from a shorter bar viewed at $\psi = 0^\circ$ just from photometry.

(It might be easier at lower inclinations $i < 90^\circ$).

Kinematics / dynamical modelling should help?

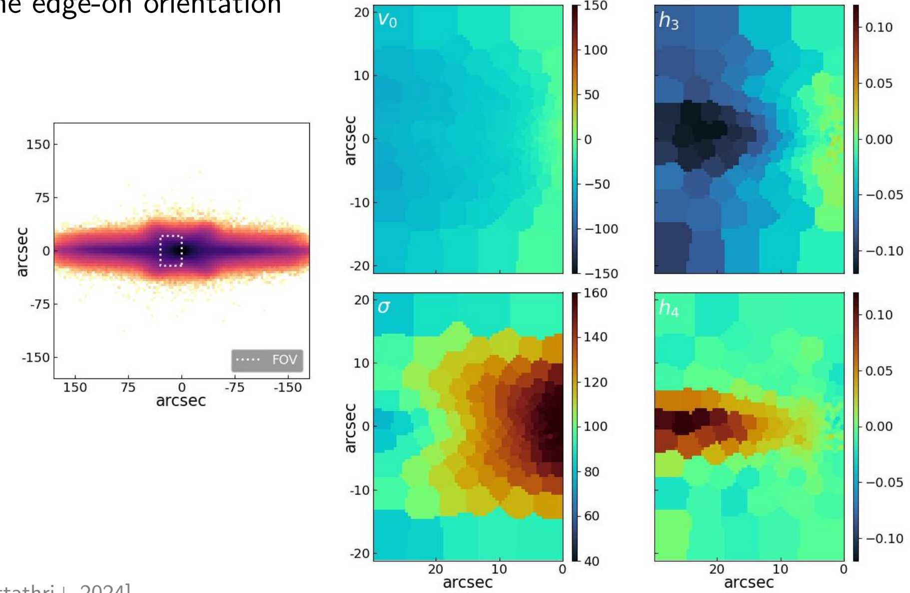




[Dattathri+ 2024]

Schwarzchild modelling of deprojected bars

MUSE-like kinematic maps ($1'$ FoV) of a Milky Way-like galaxy at $D = 20$ Mpc in the edge-on orientation

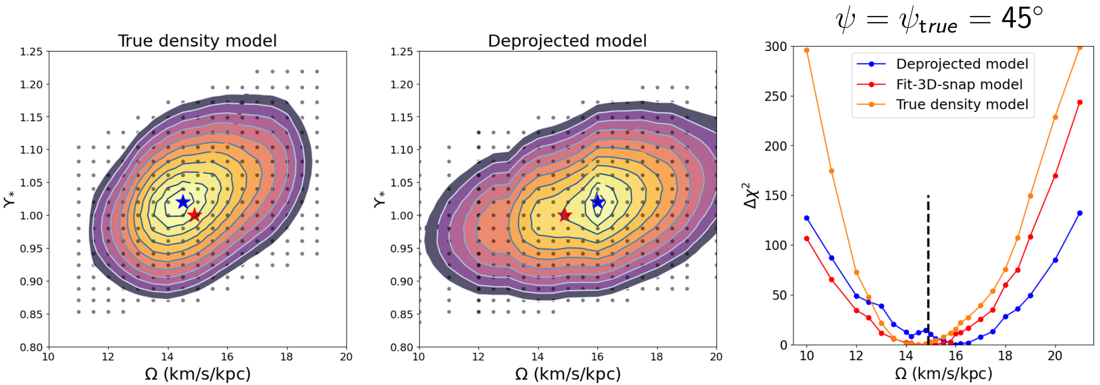


[Dattathri+ 2024]

Recovery of bar pattern speed

Ω is recovered almost perfectly if the true 3d density is used, or to within 10% if the deprojected density is used.

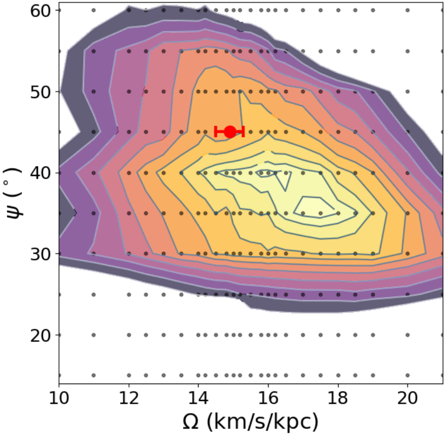
This is for the most challenging edge-on orientation, where the Tremaine–Weinberg method is not applicable!

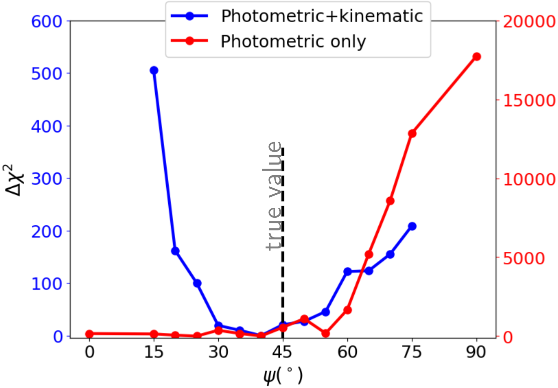


[Dattathri+ 2024]

Recovery of bar orientation

Bar orientation is also constrained much better than from pure photometry

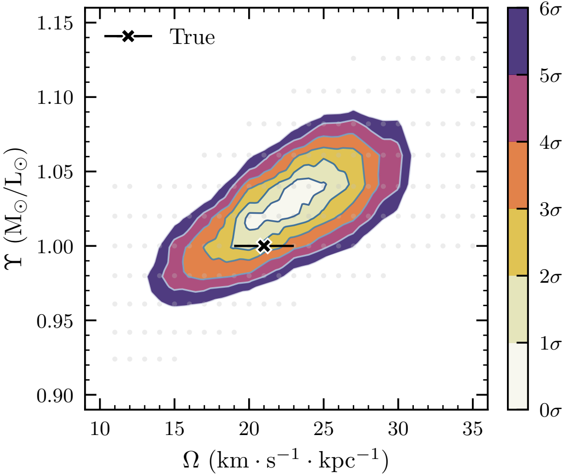




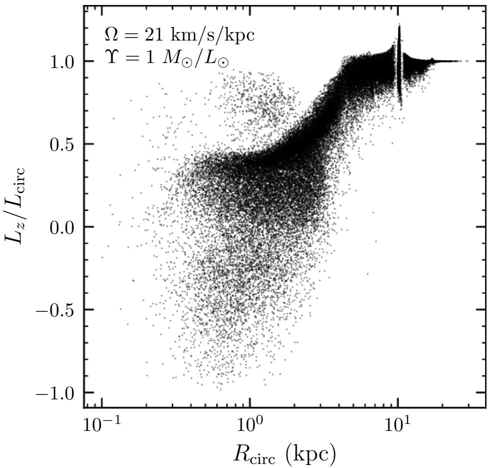
[Dattathri+ 2024]

Extending the test to moderate inclinations ($i = 30^\circ$)

recovery of pattern speed and M/L



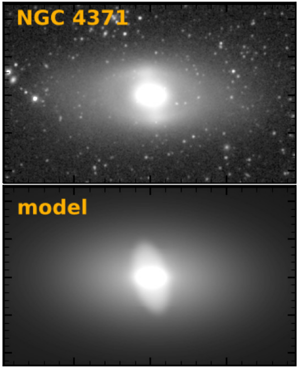
orbital circularity distribution

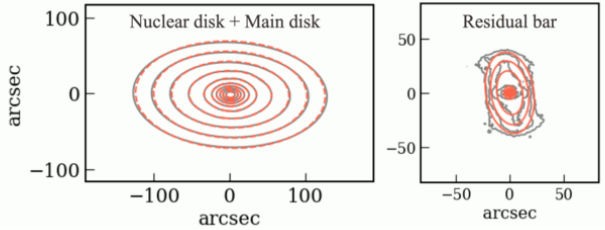


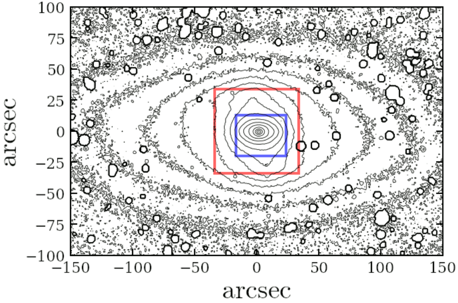
[credit: Callum Bloor]

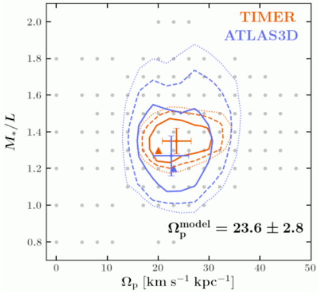
Another implementation of Schwarzschild's method for bars

by Tahmasebzadeh+ 2021, 2022, based on the triaxial code DYNAMITE;
use MGE for deprojection, separately for the main disc and the bar





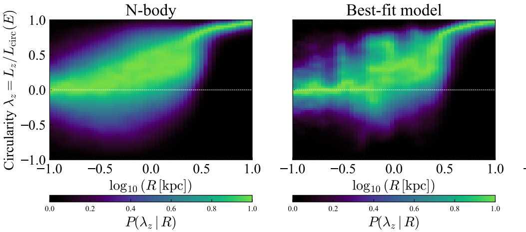


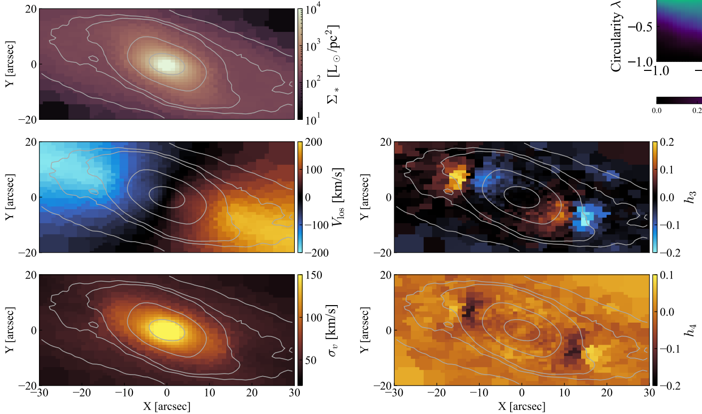


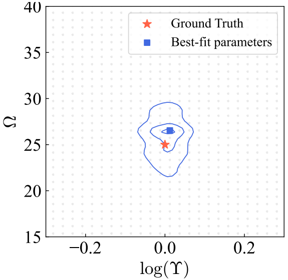
[Tahmasebzadeh+ 2024]

Yet another variant of Schwarzschild's method

SchwarMAX [Zhang+ 2026, arXiv:2607.00117] – written from scratch in JAX, uses forward-modelling of both density profile (with parametric family of models from Dehnen & Aly 2023) and kinematics – no separate photometric fitting stage; model evaluation on GPU takes $\mathcal{O}(1)$ seconds.







Summary of modelling approaches

<table>
<tr><th></th><th>“classical” Schwarzschild</th><th>(unnamed) [Dattathri+ 2024]</th><th>SchwarMAX</th><th>DF-based</th></tr>
<tr><td>DF</td><td colspan="3">non-parametric $f(\mathcal{I}) = \sum_{k=1}^{N_\mathrm{orbits}} w_k\, \delta(\mathcal{I} - \mathcal{I}_k)$</td><td>parametric $f(\mathcal{I})$</td></tr>
<tr><td>3d density</td><td>deprojected</td><td colspan="2">parametric $\rho(\mathbf{x})$</td><td>from DF: $\rho(\mathbf{x}) = \iiint f(\mathbf{x},\mathbf{v})\,\mathrm{d}^3\mathbf{v}$</td></tr>
<tr><td>surface density</td><td>parametric (MGE, GalFit)</td><td colspan="3">from ρ (integrated along the line of sight)</td></tr>
<tr><td>photometry is fitted</td><td colspan="2">separately (beforehand)</td><td colspan="2">simultaneously with kinematics</td></tr>
</table>

[alternatives perhaps worth exploring]