Is chaos relevant for galactic dynamics?

Eugene Vasiliev

IFPU focus week – Chaos and nonlinearity in dynamical astronomy

Trieste, 26 February 2026

Chaos in the N-body problem

ON THE SENSITIVITY OF THE N-BODY PROBLEM TO SMALL CHANGES IN INITIAL CONDITIONS

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Received 1990 October 29; accepted 1990 December 4

AVERAGE e-FOLDING TIMES

N	t_*/t_{cr} [a]
16	0.89 ± 0.05
30	0.77 ± 0.07
55	0.74 ± 0.06
100	0.72 ± 0.06
185	0.69 ± 0.05
340	0.69 ± 0.02

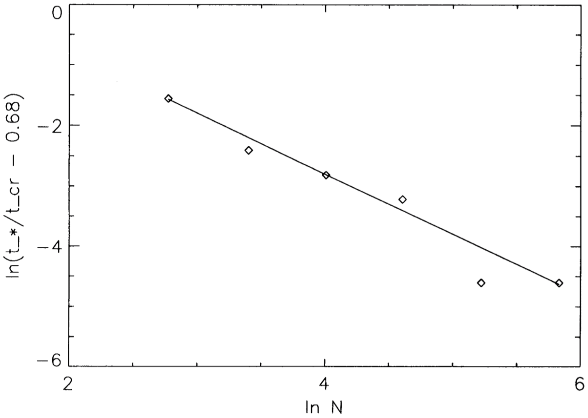

FIG. 4.—Mean e-folding time t_* as a function of the number N of particles in the simulation. The solid line represents the function $\ln(t_*/t_{cr} - 0.68) = \ln(3.32/N)$.

Chaos in the N-body problem

INSTABILITY OF THE GRAVITATIONAL N-BODY PROBLEM IN THE LARGE-N LIMIT

MARC HEMSENDORF AND DAVID MERRITT
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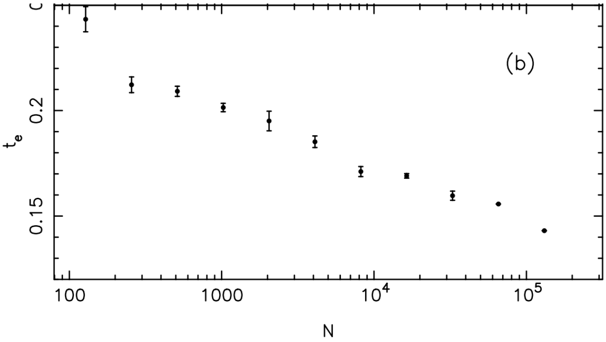
Received 2002 May 31; accepted 2002 July 19

ABSTRACT

We use a systolic N-body algorithm to evaluate the linear stability of the gravitational N-body problem for N up to 1.3×10^5, 2 orders of magnitude greater than in previous experiments. For the first time, a clear N-dependence of the perturbation growth rate is seen, $\mu_e \sim \ln N$. The e-folding time for $N = 10^5$ is roughly $1/20$ of a crossing time.



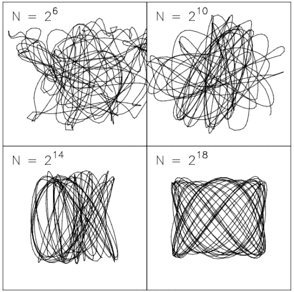
Chaos in the N-body problem

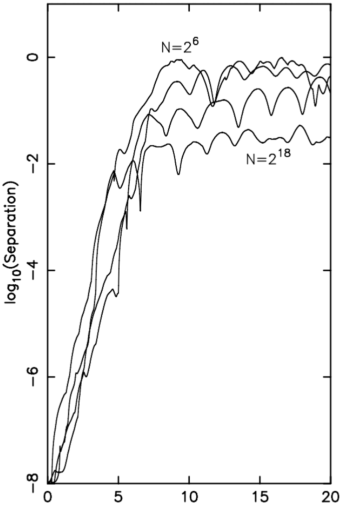
Orbital instability and relaxation in stellar systems

Monica Valluri[1,2] and David Merritt[1]

ABSTRACT
We review recent progress in understanding the role of chaos in influencing the structure and evolution of galaxies. The orbits of stars in galaxies are generically chaotic: the chaotic behavior arises in part from the intrinsically grainy nature of a potential that is composed of point masses. Even if the potential is assumed to be smooth, however, much of the phase space of non-axisymmetric galaxies is chaotic due to the presence of central density cusps or black holes. The chaotic nature of orbits implies that perturbations will grow exponentially and this in turn is expected to result in a diffusion in phase space. We show that the degree of orbital evolution is not well predicted by the growth rate of infinitesimal perturbations, i.e. by the Liapunov exponent. A more useful criterion is whether perturbations continue to grow exponentially until their scale is of order the size of the system. We illustrate these ideas in a potential consisting of N fixed point masses. Liapunov exponents are large for all values of N, but orbits become increasingly regular in their behavior as N increases; the reason is that the exponential divergence saturates at smaller and smaller distances as N is increased.

arXiv:9909403





Chaos in the N-body problem

Chaos and the continuum limit in the gravitational N-body problem: Integrable potentials

Henry E. Kandrup*

Ioannis V. Sideris†

(Received 26 March 2001; revised manuscript received 19 June 2001; published 18 October 2001)

This paper summarizes a numerical investigation of the statistical properties of orbits evolved in ''frozen,'' time-independent N-body realizations of smooth, time-independent density distributions corresponding to integrable potentials, allowing for $10^{2.5} \leq N \leq 10^{5.5}$. Two principal conclusions were reached: (1) In agreement with recent work by Valluri and Merritt, one finds that, in the limit of a nearly ''unsoftened'' two-body kernel, i.e., $V(r) \propto (r^2+\epsilon^2)^{-1/2}$ for $\epsilon \rightarrow 0$, the value of the largest Lyapunov exponent χ does *not* decrease systematically with increasing N, so that, viewed in terms of the sensitivity of individual orbits to small changes in initial conditions, there is no sense in which chaos ''turns off'' for large N. However, it is clear that, for any finite ϵ, χ *will* tend to zero for sufficiently large N. (2) Even though χ does not decrease for an unsoftened kernel, there is a clear, quantifiable sense in which, as N increases, chaotic orbits in the frozen-N systems remain ''close to'' integrable characteristics in the smooth potential for progressively longer times.

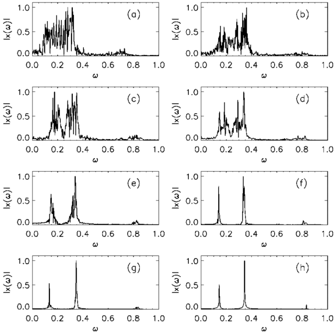


FIG. 11. (a) The Fourier transformed $|x(\omega)|$ for one frozen-N

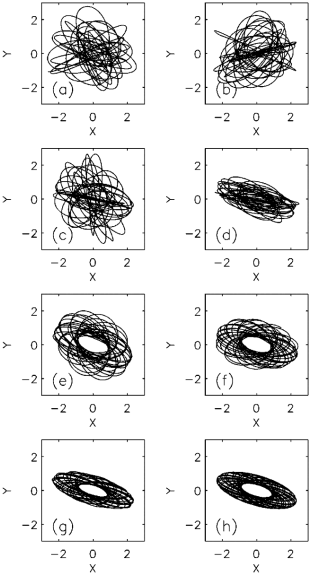


FIG. 10. The x-y projection of representative frozen-N orbits generated from the same initial condition, evolved for $t=25t_D$ with $\epsilon=10^{-5}$. (a) $N=316$. (b) $N=1000$. (c) $N=3163$. (d) $N=10\,000$. (e) $N=31\,623$. (f) $N=100\,000$. (g) $N=316\,228$. (h) The x-y projection of the same initial condition evolved in the smooth potential.

Chaos in the N-body problem

Chaos and the continuum limit in the gravitational N-body problem: Integrable potentials

Henry E. Kandrup*

Ioannis V. Sideris†

(Received 26 March 2001; revised manuscript received 19 June 2001; published 18 October 2001)

This paper summarizes a numerical investigation of the statistical properties of orbits evolved in ‘‘frozen,’’ time-independent N-body realizations of smooth, time-independent density distributions corresponding to integrable potentials, allowing for $10^{2.5} \leqslant N \leqslant 10^{5.5}$. Two principal conclusions were reached: (1) In agreement with recent work by Valluri and Merritt, one finds that, in the limit of a nearly ‘‘unsoftened’’ two-body kernel, i.e., $V(r) \propto (r^2+\epsilon^2)^{-1/2}$ for $\epsilon \rightarrow 0$, the value of the largest Lyapunov exponent χ does *not* decrease systematically with increasing N, so that, viewed in terms of the sensitivity of individual orbits to small changes in initial conditions, there is no sense in which chaos ‘‘turns off’’ for large N. However, it is clear that, for any finite ϵ, χ *will* tend to zero for sufficiently large N. (2) Even though χ does not decrease for an unsoftened kernel, there is a clear, quantifiable sense in which, as N increases, chaotic orbits in the frozen-N systems remain ‘‘close to’’ integrable characteristics in the smooth potential for progressively longer times.

Chaos and the continuum limit in the gravitational N-body problem. II. Nonintegrable potentials

Ioannis V. Sideris*

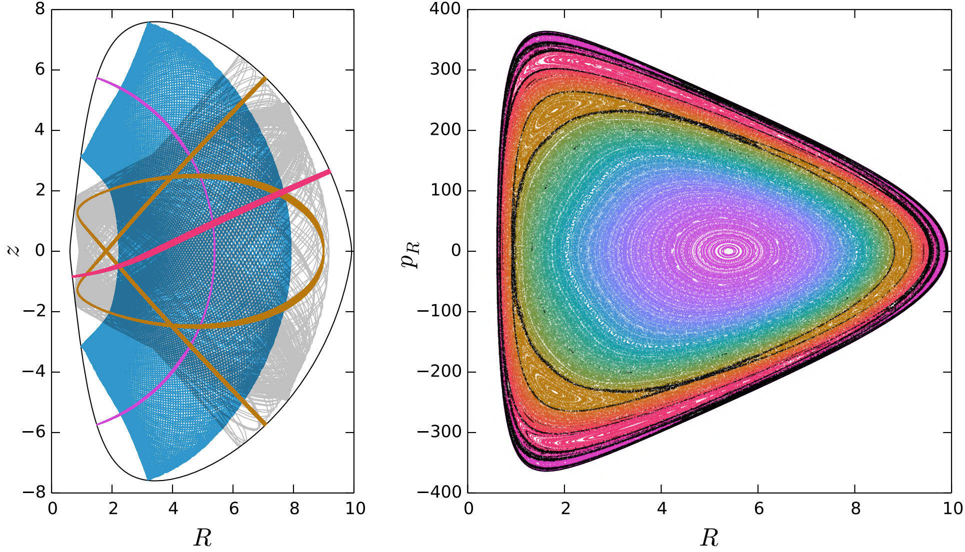
Henry E. Kandrup†

(Received 3 December 2001; revised manuscript received 26 February 2002; published 11 June 2002)

This paper continues a numerical investigation of the statistical properties of ‘‘frozen-N orbits,’’ i.e., orbits evolved in frozen, time-independent N-body realizations of smooth density distributions ρ corresponding to both integrable and nonintegrable potentials, allowing for $10^{2.5} \leqslant N \leqslant 10^{5.5}$. The focus is on distinguishing between, and quantifying, the effects of graininess on initial conditions corresponding, in the continuum limit, to regular and chaotic orbits. Ordinary Lyapunov exponents χ do not provide a useful diagnostic for distinguishing between regular and chaotic behavior. Frozen-N orbits corresponding in the continuum limit to both regular and chaotic characteristics have large positive χ even though, for large N, the ‘‘regular’’ frozen-N orbits closely resemble regular characteristics in the smooth potential. Alternatively, viewed macroscopically, both regular and ‘‘chaotic’’ frozen-N orbits diverge as a power law in time from smooth orbits with the same initial condition. However, convergence towards the continuum limit is much slower for chaotic orbits. For regular orbits, the time scale associated with this divergence $t_G \sim N^{1/2} t_D$, with t_D a characteristic dynamical, or crossing, time; for chaotic orbits $t_G \sim (\ln N) t_D$. For $N > 10^3 - 10^4$, clear distinctions exist between the phase

Chaos in smooth potentials

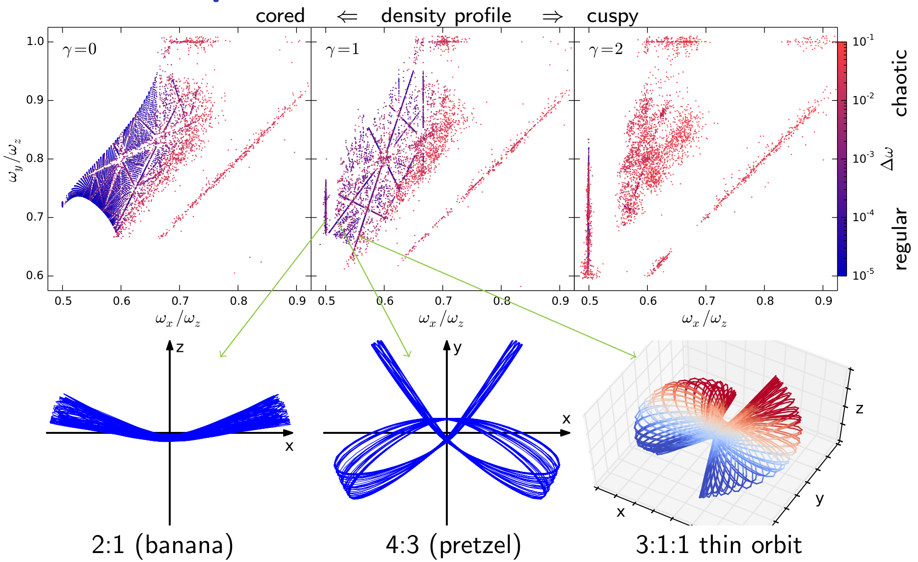
2d Hamiltonian: motion in the meridional plane of an axisymmetric galaxy (fixed E and L_z)



Chaos in smooth potentials

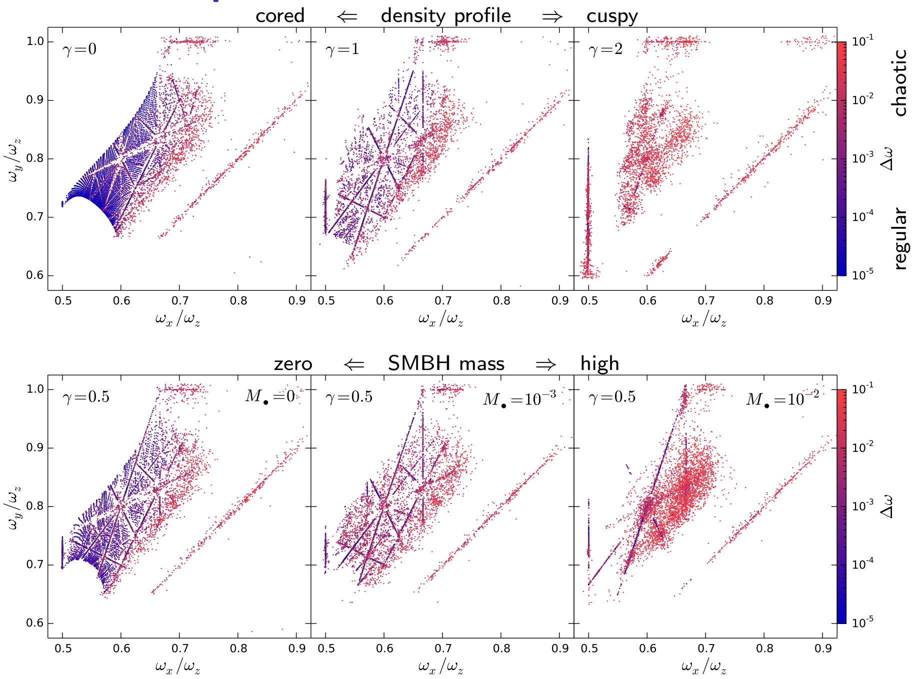
Friedrich Hundertwasser – Grass for those who cry

Chaos in smooth potentials

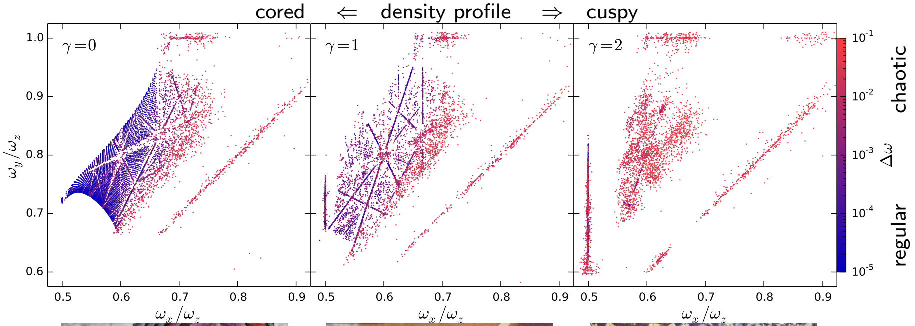


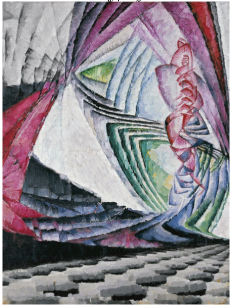
Frequency maps [Papaphilippou & Laskar 1998; Wachlin & Ferraz-Mello 1998; Merritt & Valluri 1999] of box orbits in triaxial Dehnen models at the half-mass radius coloured by the frequency diffusion rate [Laskar 1993]

Chaos in smooth potentials

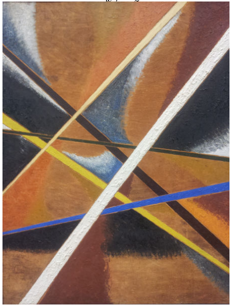


Chaos in smooth potentials

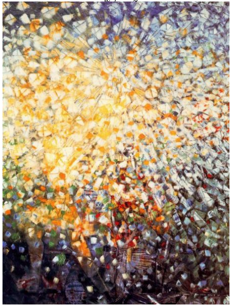




František Kupka – Localization of graphic motifs



Lyubov Popova – Futuristic composition



Max Ernst – 33 little girls chasing butterflies

Chaos-driven evolution of galactic shape

SELF-CONSISTENT GRAVITATIONAL CHAOS

David Merritt and Monica Valluri

Rutgers University, New Brunswick, NJ

Abstract: The motion of stars in the gravitational potential of a triaxial galaxy is generically chaotic. However, the timescale over which the chaos manifests itself in the orbital motion is a strong function of the degree of central concentration of the galaxy. Here, chaotic diffusion rates are presented for orbits in triaxial models with a range of central density slopes and nuclear black-hole masses. Typical diffusion times are found to be less than a galaxy lifetime in triaxial models where the density increases more rapidly than $\sim r^{-1}$ at the center, or which contain black holes with masses that exceed $\sim 0.1\%$ of the galaxy mass. When the mass of a central black hole exceeds roughly 0.02 M_{gal}, there is a transition to global stochasticity and the galaxy evolves to an axisymmetric shape in little more than a crossing time. This rapid evolution may provide a negative feedback mechanism that limits the mass of nuclear black holes to a few percent of the stellar mass of a galaxy.

Chaos-driven evolution of galactic shape

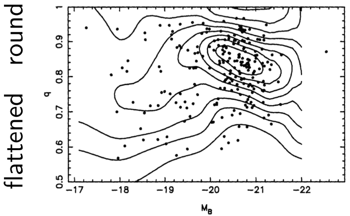
EVIDENCE FROM INTRINSIC SHAPES FOR TWO FAMILIES OF ELLIPTICAL GALAXIES

B. TREMBLAY AND D. MERRITT

Received 1996 January 19; revised 1996 February 22

ABSTRACT

Bright elliptical galaxies have a markedly different distribution of Hubble types than faint ellipticals; the division occurs near $M_B = -20$ and bright ellipticals are rounder on average.



fainter
oblate
fast rotators

brighter
triaxial
slow rotators

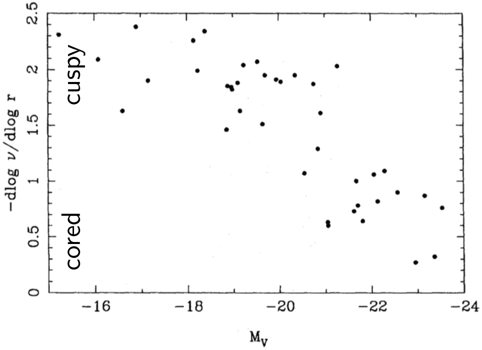
THE CENTERS OF EARLY-TYPE GALAXIES WITH *HST*. III. NON-PARAMETRIC RECOVERY OF STELLAR LUMINOSITY DISTRIBUTIONS

KARL GEBHARDT AND Nukers

Received 1995 November 3; revised 1996 March 8

ABSTRACT

We have non-parametrically determined the luminosity density profiles and their logarithmic slopes for 42 early-type galaxies observed with *HST*. Assuming that the isodensity contours are spheroidal, then the luminosity density is uniquely determined from the surface brightness data through the Abel equation. For nearly all the galaxies in our sample, the logarithmic slope of the luminosity density ($S = d \log \nu / d \log r$) measured at 0.1″ (the innermost reliable measurement with the uncorrected *HST*) is significantly different from zero; i.e., most elliptical galaxies have cusps. There are only two galaxies for which an analytic core ($S \rightarrow 0$) cannot be excluded. The distribution of logarithmic slopes at 0.1″ appears to be bimodal, confirming the conclusion of Lauer *et al.* [AJ, 110, 2622 (1995)] that early-type galaxies can be divided into two types based on their surface brightness profiles; i.e., those with cuspy cores and those whose steep power-law profiles continue essentially unchanged in to the resolution limit. The peaks in the slope distribution occur at $S = -0.8$ and -1.9. More than half of the galaxies have slopes steeper than -1.0. Taken together with the recent theoretical work of Merritt and Fridman, these results suggest that many (and maybe most) elliptical galaxies are either nearly axisymmetric or spherical near the center, or slowly evolve due to the influence of stochastic orbits.



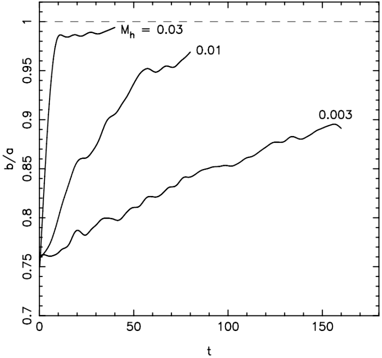
Chaos-driven evolution of black hole feeding

DYNAMICAL EVOLUTION OF ELLIPTICAL GALAXIES WITH CENTRAL SINGULARITIES

David Merritt and Gerald D. Quinlan
Department of Physics and Astronomy, Rutgers University, New Brunswick, NJ 08855
Received 1997 August 8; accepted 1997 December 23

ABSTRACT

We study the effect of a massive central singularity on the structure of a triaxial galaxy using N-body simulations. Starting from a single initial model, we grow black holes with various final masses M_h and at various rates, ranging from impulsive to adiabatic. In all cases, the galaxy achieves a final shape that is nearly spherical at the center and close to axisymmetric throughout. However, the rate of change of the galaxy's shape depends strongly on the ratio M_h/M_g of black hole mass to galaxy mass. When $M_h/M_g \lesssim 0.3\%$, the galaxy evolves in shape on a timescale that exceeds 10^2 orbital periods, or roughly a galaxy lifetime. When $M_h/M_g \gtrsim 2.5\%$, the galaxy becomes axisymmetric in little more than a crossing time. We propose that the rapid evolution toward axisymmetric shapes that occurs when $M_h/M_g \gtrsim 2.5\%$ provides a negative-feedback mechanism that limits the mass of central black holes by cutting off their supply of fuel.



Chaos in stellar streams

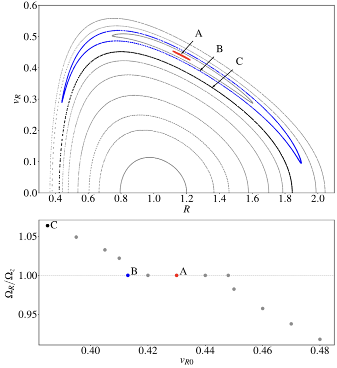
Separatrix divergence of stellar streams in galactic potentials

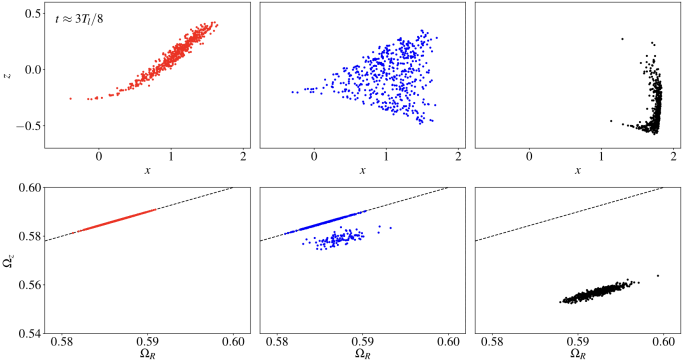
Tomer D. Yavetz ,[1★] Kathryn V. Johnston,[1,2] Sarah Pearson ,[2] Adrian M. Price-Whelan [2,3]
and Martin D. Weinberg[4]

Accepted 2020 November 16. Received 2020 October 22; in original form 2020 March 5

ABSTRACT

Flattened axisymmetric galactic potentials are known to host minor orbit families surrounding orbits with commensurable frequencies. The behaviour of orbits that belong to these orbit families is fundamentally different than that of typical orbits with non-commensurable frequencies. We investigate the evolution of stellar streams on orbits near the boundaries between orbit families (separatrices) in a flattened axisymmetric potential. We demonstrate that the separatrix divides these streams into two groups of stars that belong to two different orbit families, and that as a result, these streams diffuse more rapidly than streams that evolve elsewhere in the potential. We utilize Hamiltonian perturbation theory to estimate both the time-scale of this





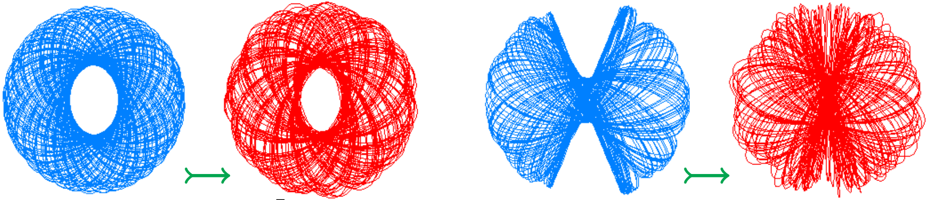
Relation between smooth-potential and N-body chaoses

CHAOS AND NOISE IN GALACTIC POTENTIALS

SALMAN HABIB,[1] HENRY E. KANDRUP,[2,3] AND M. ELAINE MAHON[2]
Received 1994 December 5; accepted 1996 November 7

ABSTRACT

This paper summarizes an investigation of the effects of weak friction and noise in time-independent, nonintegrable two-dimensional potentials that admit both regular and stochastic orbits. The aim is to understand the qualitative effects of internal and external irregularities associated with discreteness effects or couplings to an external environment, which stars in any real galaxy must experience. It is found that these irregularities can be important already on timescales much shorter than the natural relaxation timescale t_R associated with two-body relaxation. In particular, for stochastic orbits friction and noise result in an exponential divergence from the unperturbed Hamiltonian trajectory, at a rate set by the value of the local Lyapunov exponent, which persists even for relatively large deviations from the unperturbed trajectory. Friction and noise can also have significant effects on the *statistical* properties of ensembles of stochastic orbits. Stochastic orbits may be divided into two classes, confined or sticky stochastic orbits which are trapped near islands of regularity, and unconfined or filling stochastic orbits that travel unimpeded throughout a stochastic sea. In the absence of friction and noise, transitions between confined and filling stochastic orbits are very slow. However, even very weak friction and noise can drastically accelerate such transitions, leading to an approach toward a statistical equilibrium on timescales $\ll t_R$. In the two-dimensional models studied in this paper, there are cases for which t_R exceeds



noise ($N \simeq 10^7$)

Relation between smooth-potential and N-body chaoses

Phase-space transport in cuspy triaxial potentials: can they be used to construct self-consistent equilibria?

Christos Siopis[1,2★] and Henry E. Kandrup[1,2,3,4★]

Accepted 2000 May 8. Received 2000 April 20; in original form 2000 February 8

ABSTRACT

This paper focuses on the statistical properties of chaotic orbit ensembles evolved in triaxial generalizations of the Dehnen potential which have been proposed recently to model realistic ellipticals that have a strong density cusp and manifest significant deviations from axisymmetry. Allowance is made for a possible supermassive black hole, as well as low-amplitude friction, noise, and periodic driving which can mimic irregularities associated with discreteness effects and/or an external environment. The chaos exhibited by these potentials is quantified by determining (1) how the relative number of chaotic orbits depends on the steepness of the cusp, as probed by γ, the power-law exponent with which density diverges, and $M_{\rm BH}$, the black hole mass, (2) how the size of the largest Lyapunov exponent varies with γ and $M_{\rm BH}$, and (3) the extent to which Arnold webs significantly impede phase-space transport, both with and without perturbations. The most important conclusions dynamically are (1) that, in the absence of irregularities, chaotic orbits tend to be *extremely* 'sticky', so that different pieces of the same chaotic orbit can behave very differently for times $\sim 10000 t_{\rm D}$ or more, but (2) that even very low-amplitude perturbations can prove efficient in erasing many – albeit not all – of these differences. The implications of these facts are discussed both for the structure and evolution of real galaxies and for the possibility of constructing approximate near-equilibrium models using Schwarzschild's method. For example, when trying to use Schwarzschild's method to construct model galaxies containing significant numbers of chaotic orbits, it seems advantageous to build libraries with chaotic orbits evolved in the presence of low-amplitude friction and noise, since such noisy orbits are more likely to represent reasonable approximations to time-independent building blocks.

Simulating the effect of two-body relaxation probabilistically

velocity perturbations (drift & diffusion) [Rosenbluth+ 1957, Spitzer 1970s]:

$$\Delta v_{\|} = \langle \Delta v_{\|} \rangle \Delta t + \zeta_1 \sqrt{\langle \Delta v_{\|}^2 \rangle \Delta t}, \quad \Delta v_{\perp} = \zeta_2 \sqrt{\langle \Delta v_{\perp}^2 \rangle \Delta t}\,, \qquad \zeta_1, \zeta_2 \sim \mathcal{N}(0,1)$$

$$v\langle \Delta v_{\|} \rangle = -\left(1 + \tfrac{m}{m_\star}\right) I_{1/2}, \quad \langle \Delta v_{\|}^2 \rangle = \tfrac{2}{3}\left(I_0 + I_{3/2}\right), \quad \langle \Delta v_{\perp}^2 \rangle = \tfrac{2}{3}\left(2I_0 + 3I_{1/2} - I_{3/2}\right)$$

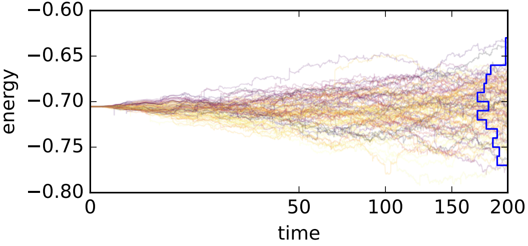
$$I_0 \equiv \Gamma \int_E^0 dE'\, f(E'), \quad I_{n/2} \equiv \Gamma \int_{\Phi(r)}^E dE'\, f(E') \left(\frac{E' - \Phi(r)}{E - \Phi(r)}\right)^{n/2}$$

$$\Gamma \equiv 16\pi^2 G^2 M_{\rm tot} \times (N_\star^{-1} \ln \Lambda)$$

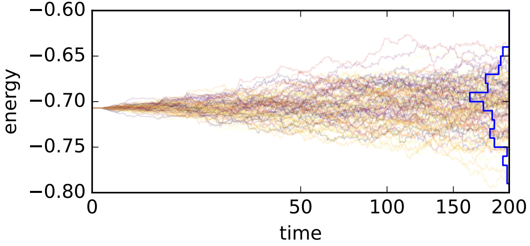
adjustable amount of relaxation

Energy diffusion of two ensembles of trajectories in a Plummer potential plus...

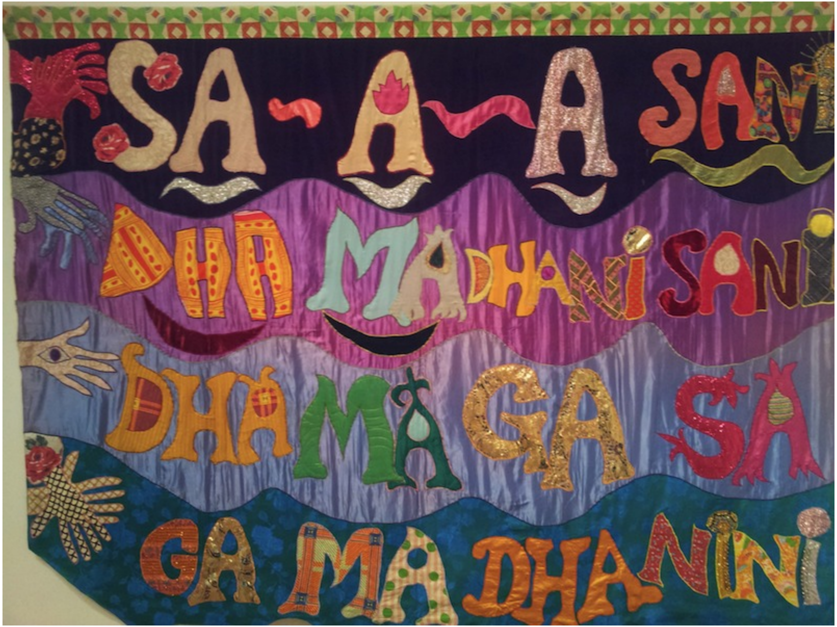
10^4 moving point masses (10% of total mass)



random velocity kicks after each timestep



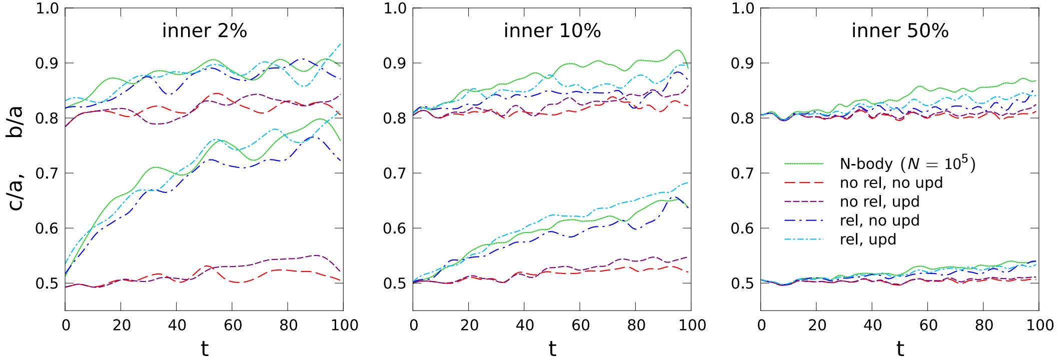
– relaxation in any geometry [Vasiliev 2015]



Moki Cherry – Raga

Relaxation + chaotic diffusion in triaxial galaxies

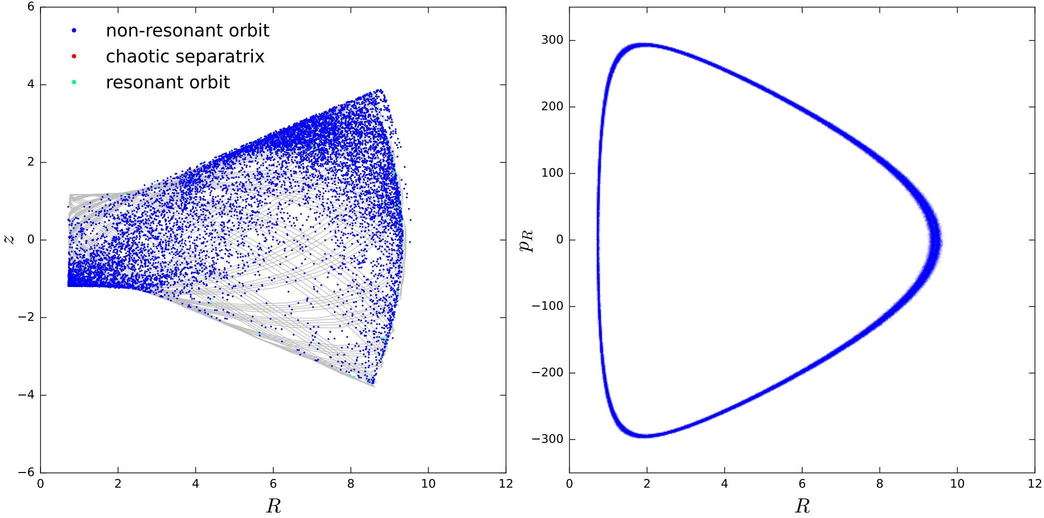
- Without relaxation, the shape of the system remains stable
- Addition of relaxation accelerates diffusion of chaotic orbits and leads to a gradual loss of triaxiality
- Good agreement between Monte Carlo and N-body codes for the same amount of relaxation



triaxial $\gamma = 1$ Dehnen model with $M_\bullet = 10^{-2}$ [Vasiliev 2015; Hamilton & Vasiliev, unpublished]

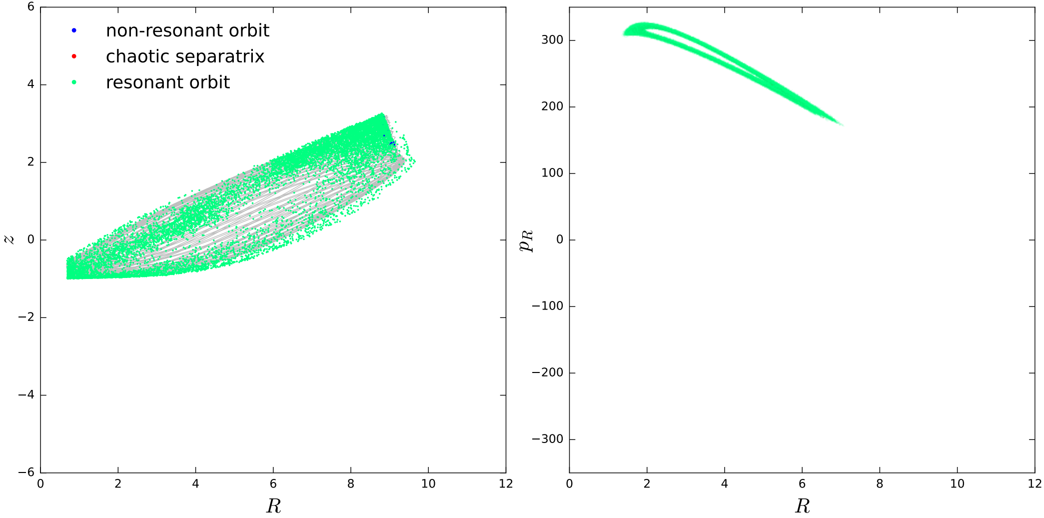
Relaxation + chaotic separatrix divergence in stellar streams

tidal stream from a cluster on a non-resonant tube orbit



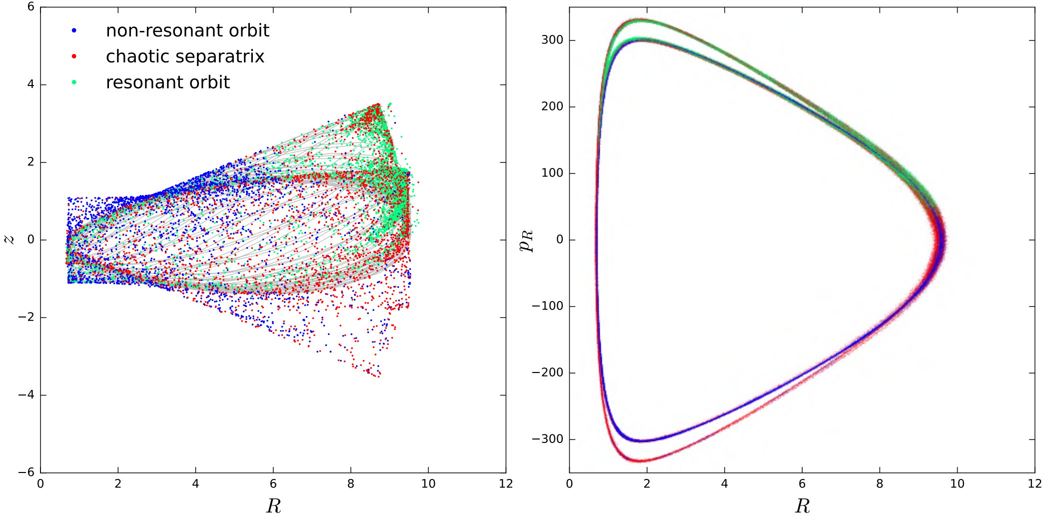
Relaxation + chaotic separatrix divergence in stellar streams

tidal stream from a cluster on a resonant "saucer" orbit



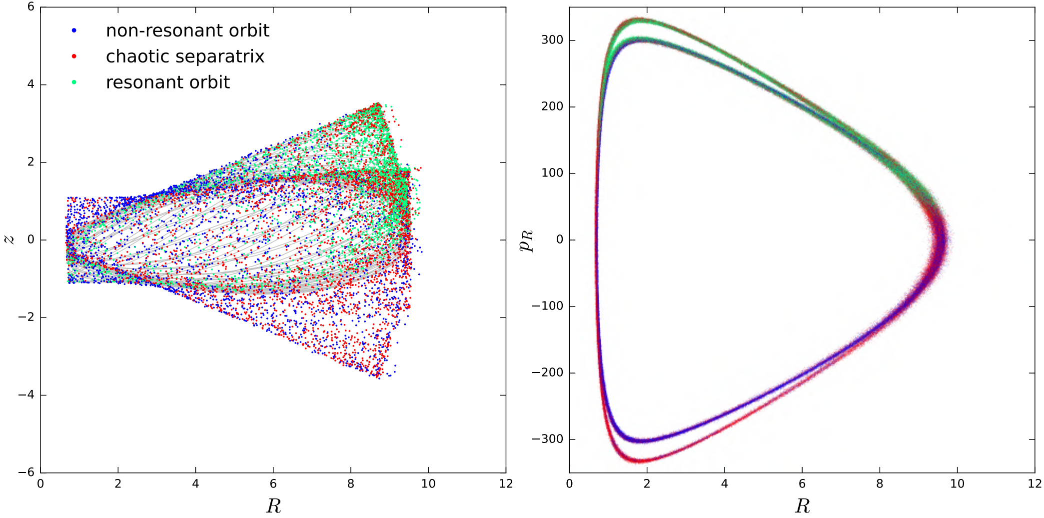
Relaxation + chaotic separatrix divergence in stellar streams

tidal stream from a cluster on a chaotic separatrix orbit



Relaxation + chaotic separatrix divergence in stellar streams

tidal stream from a cluster on a chaotic separatrix orbit
with added relaxation corresponding to perturbers of mass $10^4\ M_\odot$

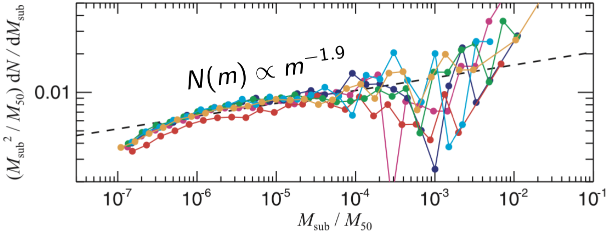


Relaxation + chaotic separatrix divergence in stellar streams

with added relaxation corresponding to perturbers of mass $10^4\ M_\odot$

why so massive? designed to mimic DM subhaloes:

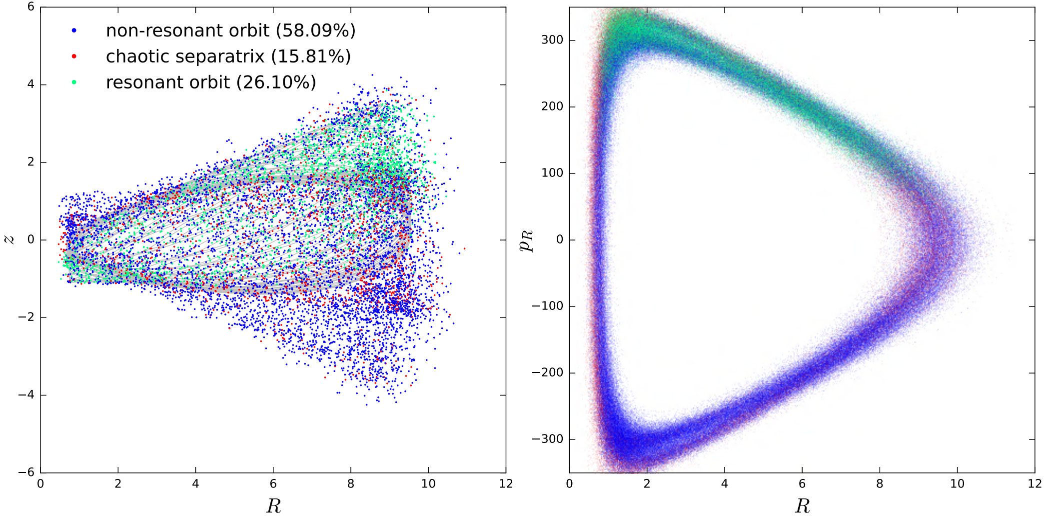
relaxation rate is $\propto \int \rho_{\rm pert}\, m_{\rm pert}\, {\rm d}m_{\rm pert}$, so is dominated by heaviest objects



Aquarius simulation [Springel+ 2008]

Relaxation + chaotic separatrix divergence in stellar streams

tidal stream from a cluster on a chaotic separatrix orbit
with added relaxation corresponding to perturbers of mass $10^6\ M_\odot$

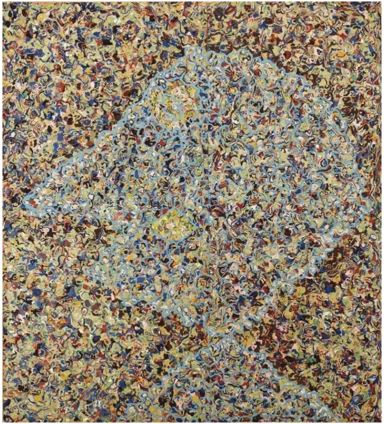


messed up too much, real streams do not look so diffuse...

Summary

So, is *chaos in smooth potentials* relevant for galactic dynamics?

- it can lead to interesting effects in galaxy evolution
- the addition of *discreteness noise* creates more chaos, but does not seem to completely erase the phenomena arising from dynamics in non-integrable potentials



Enrico Baj – Ritratto di Jackson Pollock