The limits of dynamical modelling

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Giacomo Balla – Dynamism of a dog on a leash

Equilibrium models in stellar dynamics

Distribution function of stars $f(\mathbf{x}, \mathbf{v}, t)$ satisfies [sometimes] the collisionless Boltzmann equation:

$$\frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial t} + \frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial \mathbf{x}} \mathbf{v} - \frac{\partial f(\mathbf{x}, \mathbf{v}, t)}{\partial \mathbf{v}} \frac{\partial \Phi(\mathbf{x}, t)}{\partial \mathbf{x}} = 0.$$

observational data

potential $\Leftrightarrow$ mass distribution

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potential $\Leftrightarrow$ mass distribution

observational data

not measured directly on human timescales

In order to infer anything about the potential from a time-dependent DF, need to make further assumptions about the initial state of the system, e.g., that the stars belong to a single stream or were perturbed from an equilibrium configuration in a specific way, etc.

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Steady-state assumption $\Longrightarrow$ Jeans theorem:

$$f(\mathbf{x}, \mathbf{v}) = f\big(\mathcal{I}(\mathbf{x}, \mathbf{v};\ \Phi)\big)$$

3D function (want to infer)

3D – 6D (observed)

integrals of motion ($\leq$ 3D?), e.g., $\mathcal{I} = \{E, L, \dots\}$

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When observations provide more than 3 phase-space coordinates, the potential is overconstrained! (can detect non-stationarity, but not measure it)

Self-consistent models in stellar dynamics

$$\iiint f\left(\mathcal{I}(\mathbf{x},\mathbf{v};\ \Phi)\right)\,\mathrm{d}^3\mathbf{v} \;=\; \rho(\mathbf{x}) \;=\; \frac{1}{4\pi\,G}\,\nabla^2\Phi(\mathbf{x}).$$

0th moment of the DF — Poisson eqn

Ways to solve this integro-differential equations:

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Ways to solve this integro-differential equations:

1. from ρ (and Φ, though they don't have to be related through the Poisson eqn) to f:
   - spherical systems (1d ρ,Φ): Eddington inversion (1d $f(E)$)
     or its anisotropic generalisations $f(E,L)$
     [Ossipkov 1979; Merritt 1985; Cuddeford 1991; Gerhard 1991; Wojtak+ 2008, . . .]
   - axisymmetric (2d ρ,Φ): contour integral method $\Rightarrow f(E,L_z)$ [Hunter & Qian 1993],
     but more general 3-integral DFs are possible [e.g. Dehnen & Gerhard 1993, Evans+ 1997]
   - triaxial (3d ρ,Φ): no general approach to derive 3d $f(\mathcal{I})$, but Schwarzschild
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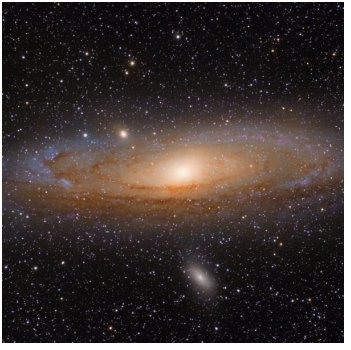
2. from f to ρ (and then Φ through Poisson eqn):
   - spherical systems: a few models defined by DFs (polytropes, King models, etc.)
   - more generally: iterative method [Kuijken & Dubinski 1995; Binney 2014]

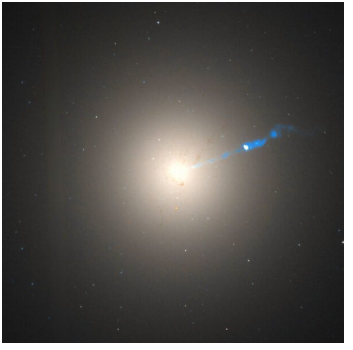
Observational data

	resolved	unresolved (into individual stars)	
X, Y sky-plane position	$+$	$+$	(surface brightness profile)
Z distance	$\mp$	$-$	
$\mu_{X,Y}$ proper motions	$\mp$	$-$	
$V_{\rm los}$ line-of-sight velocity	$\pm$	$+$	(LoS velocity distribution).
	Milky Way and LG galaxies	Galaxies outside the Local Group	

Sculptor dSph [credit: Subaru]



Andromeda [Wikipedia / B.Wesner]



M 87 [HST]

Fitting models to observations

resolved data

DF-based methods:

$$\mathcal{P}(\text{model} \mid \text{data}) \propto \prod_{i=1}^{N_\text{stars}} f\left(\mathbf{x}_i, \mathbf{v}_i;\ \boldsymbol{\alpha}\right) = \prod_{i=1}^{N_\text{stars}} f\Big(\mathcal{I}\left(\mathbf{x}_i, \mathbf{v}_i;\ \Phi(\mathbf{x};\ \boldsymbol{\beta})\right);\ \boldsymbol{\alpha}\Big),$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are the parameters of the DF and the potential, respectively, which are optimised during model fitting;
in the self-consistent case, Φ is determined by the DF, so $\boldsymbol{\beta}$ are unused.

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Jeans models:

fit for spatially binned moments $(\overline{v}, \sigma)$,

or assume a Gaussian DF $f = \rho(\mathbf{x})\mathcal{N}\big(\mathbf{v} \mid \sigma(\mathbf{x})\big)$ and proceed as above (possibly a more complicated $f(\mathbf{v})$ when using higher-order Jeans eqns)

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Orbit-based methods:

conceptually difficult... [e.g. Chaname+ 2008]

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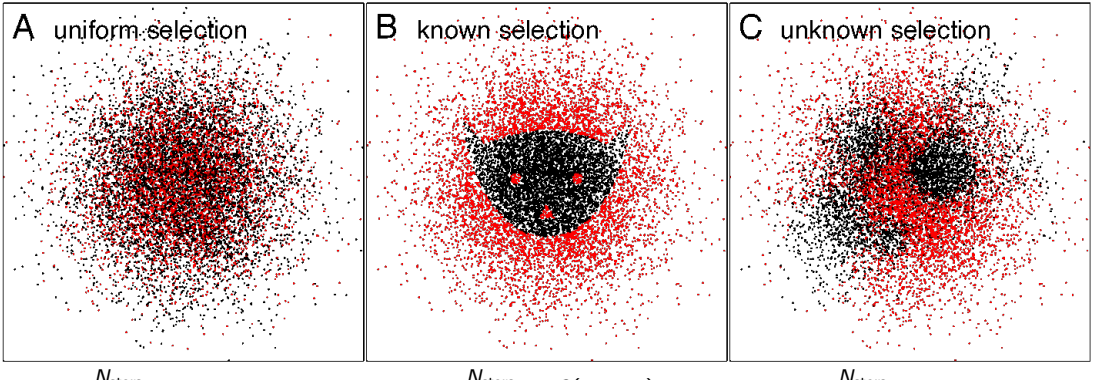
unresolved data

fit surface density $\Sigma(X, Y)$ and 3d LOSVD $f(X, Y, v_\text{los})$,
represented by velocity histograms or Gauss–Hermite moments

Episode 1: Fitting models to incomplete observations

Introduce the [spatial] selection function $S(\mathbf{x})$ – fraction of stars included in the kinematic sample



$$\ln \mathcal{L} = \sum_{i=1}^{N_{\rm stars}} \ln f(\mathbf{x}_i, \mathbf{v}_i)$$

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$$\mathcal{N} \equiv \iiint \mathrm{d}^3\mathbf{x} \iiint \mathrm{d}^3\mathbf{v}\, f(\mathbf{x}, \mathbf{v})\, S(\mathbf{x})$$

(normalization factor)

$$\ln \mathcal{L} = \sum_{i=1}^{N_{\rm stars}} \ln \mathfrak{f}(\mathbf{v}_i \mid \mathbf{x}_i),$$

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(conditional velocity distribution)

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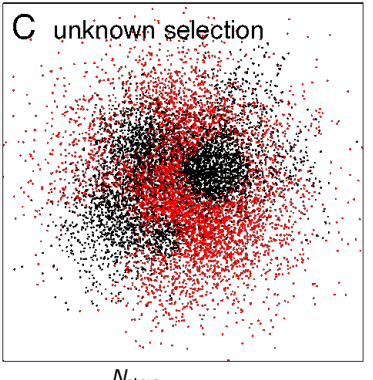
Introduce the [spatial] selection function $S(\mathbf{x})$ – fraction of stars included in the kinematic sample

A uniform selection

B known selection

C unknown selection



Can one do dynamics without knowing where [all] the tracers are?

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(conditional velocity distribution)

Toy example #1: one-dimensional vertical dynamics in a disc

Context: measure the gravitational potential $\Phi(z)$ or equivalently the force $\frac{\mathrm{d}\Phi}{\mathrm{d}z}(z)$ from stellar tracers at different heights z;
not assuming that the potential is created by stars only (non-self-consistent).

1st approach: vertical Jeans equation

$$\frac{\mathrm{d}\left(\rho\,\sigma^2\right)}{\mathrm{d}z} + \rho\,\frac{\mathrm{d}\Phi}{\mathrm{d}x} = 0$$

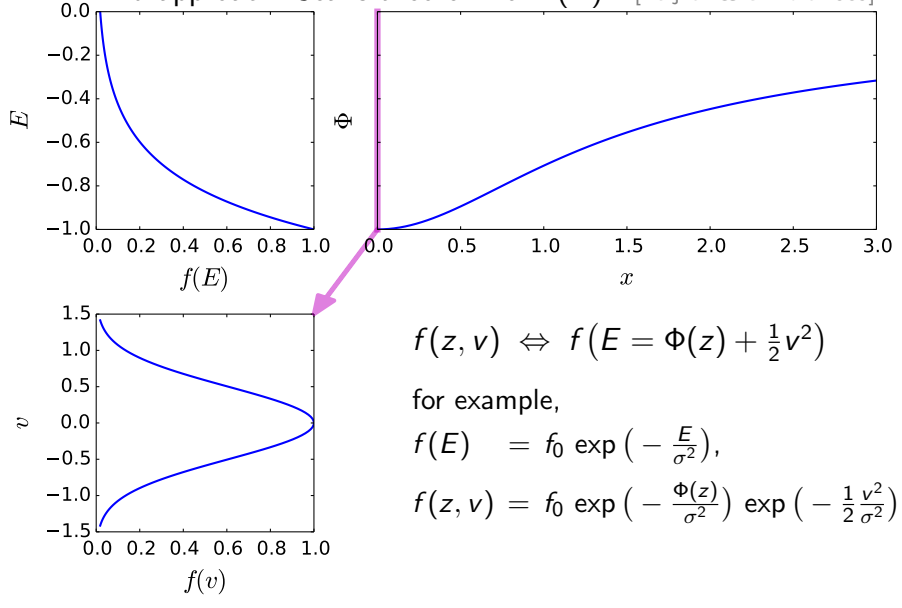
$$\frac{\mathrm{d}\ln\rho}{\mathrm{d}z} + \frac{\mathrm{d}\sigma^2}{\mathrm{d}z} = \frac{\mathrm{d}\Phi}{\mathrm{d}z}$$

typically the dominant term

workflow: $\rho(z),\ \sigma(z) \Rightarrow \Phi(z)$.

Toy example #1: one-dimensional vertical dynamics in a disc

2nd approach: Jeans theorem for $f(E)$ [Kuijken & Gilmore 1989]



$$f(z,v) \;\Leftrightarrow\; f\left(E = \Phi(z) + \tfrac{1}{2}v^2\right)$$

for example,

$$f(E) \;=\; f_0 \exp\left(-\frac{E}{\sigma^2}\right),$$

$$f(z,v) \;=\; f_0 \exp\left(-\frac{\Phi(z)}{\sigma^2}\right) \exp\left(-\frac{1}{2}\frac{v^2}{\sigma^2}\right)$$

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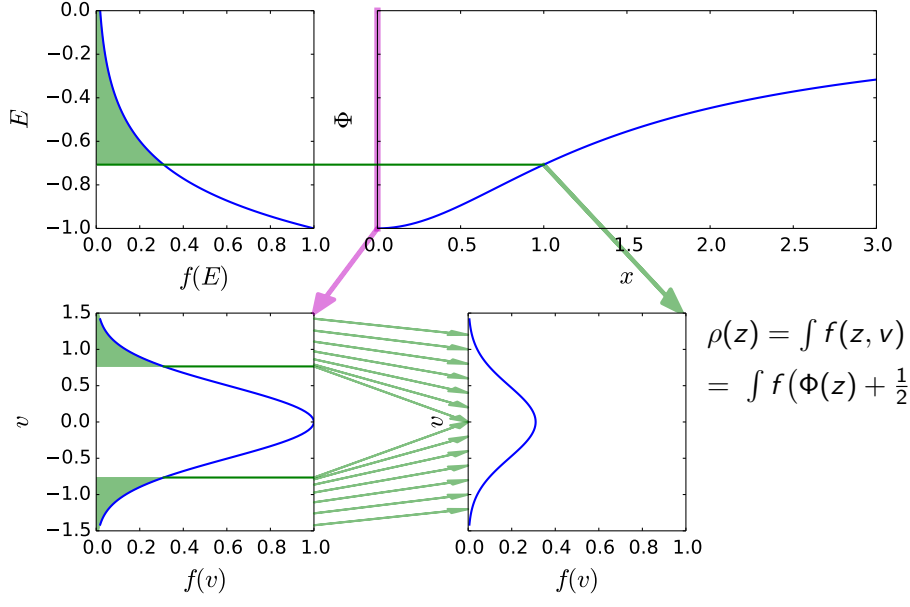

$$\rho(z) = \int f(z, v)\, \mathrm{d}v$$
$$= \int f\left(\Phi(z) + \tfrac{1}{2}v^2\right) \mathrm{d}v$$

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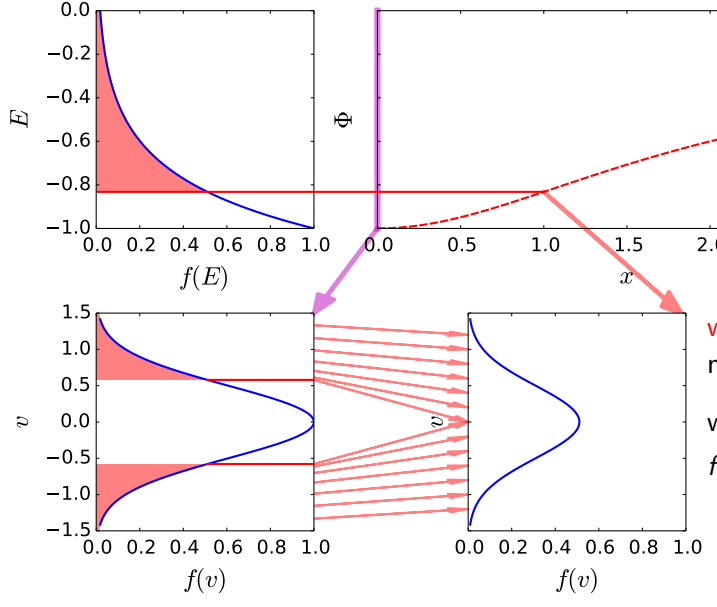


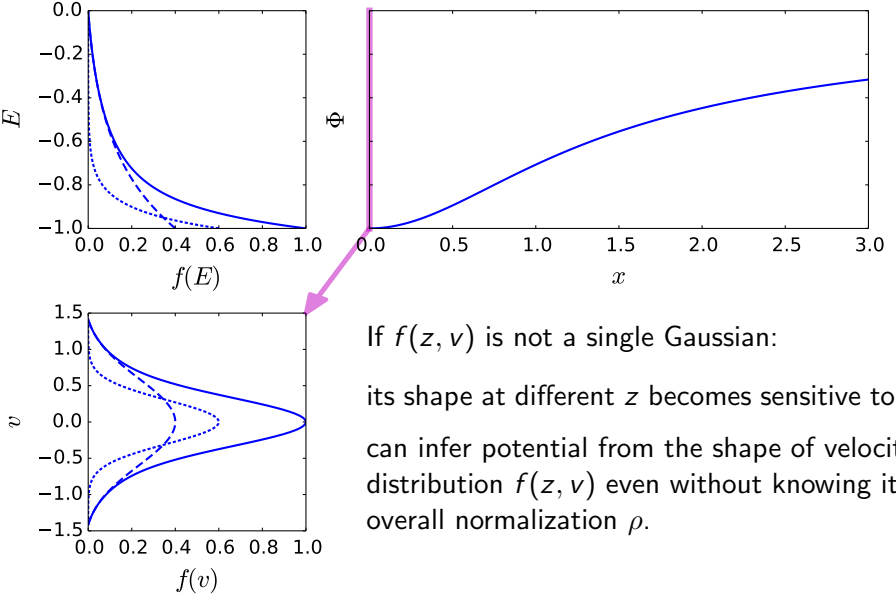
wrong potential $\Rightarrow$ mismatched $\rho(z)$.

workflow:

$f(0, v),\ \rho(z) \Rightarrow \Phi(z)$.

still necessary!

Toy example #1: one-dimensional vertical dynamics in a disc

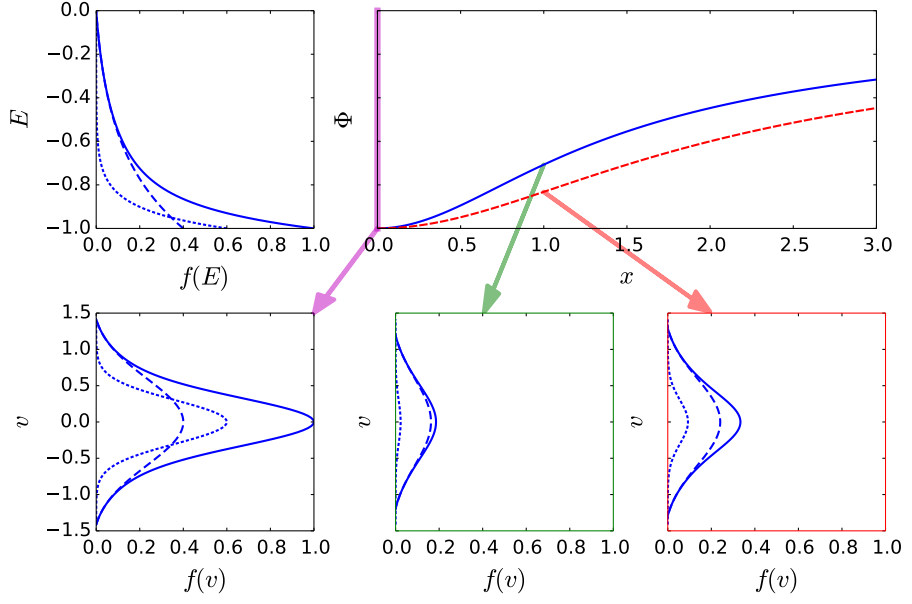


If $f(z, v)$ is not a single Gaussian:

its shape at different z becomes sensitive to Φ $\Rightarrow$

can infer potential from the shape of velocity distribution $f(z, v)$ even without knowing its overall normalization ρ.

Toy example #1: one-dimensional vertical dynamics in a disc



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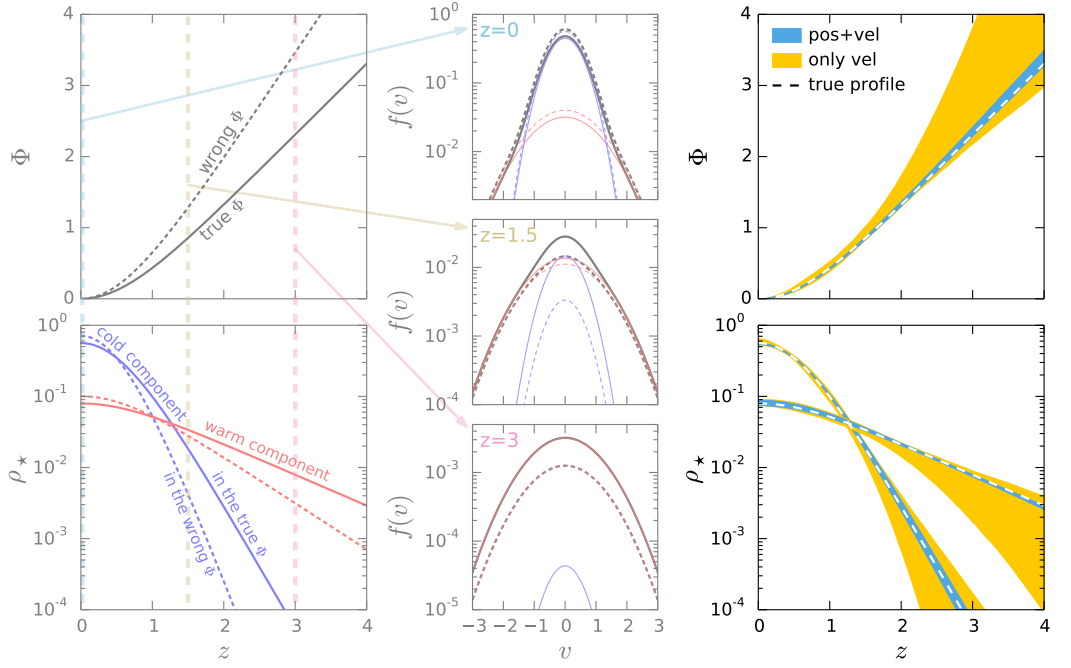

Toy example #2: spherical galaxy

	self-consistent	non-self-consistent
	$f\big(\mathcal{I}(\mathbf{x},\mathbf{v};\ \Phi);\ \boldsymbol{\alpha}\big) \Longrightarrow \rho(\mathbf{x})$ $\Phi(\mathbf{x})$	$f\big(\mathcal{I}(\mathbf{x},\mathbf{v};\ \Phi);\ \boldsymbol{\alpha}\big) \Longrightarrow \rho(\mathbf{x})$ $\Phi_{\rm ext}(\mathbf{x};\ \boldsymbol{\beta}) \Longrightarrow \rho_{\rm ext}(\mathbf{x})$
fitted parameters:	$\boldsymbol{\alpha}$ (DF)	$\boldsymbol{\alpha}$ (DF), $\boldsymbol{\beta}$ (potential)

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fitted parameters:	$\boldsymbol{\alpha}$ (DF)	$\boldsymbol{\alpha}$ (DF), $\boldsymbol{\beta}$ (potential)
known SF $\mathcal{L}_i = f(\mathbf{x}_i,\mathbf{v}_i)$		
unknown SF $\mathcal{L}_i = \dfrac{f(\mathbf{x}_i,\mathbf{v}_i)}{\rho(\mathbf{x})}$		

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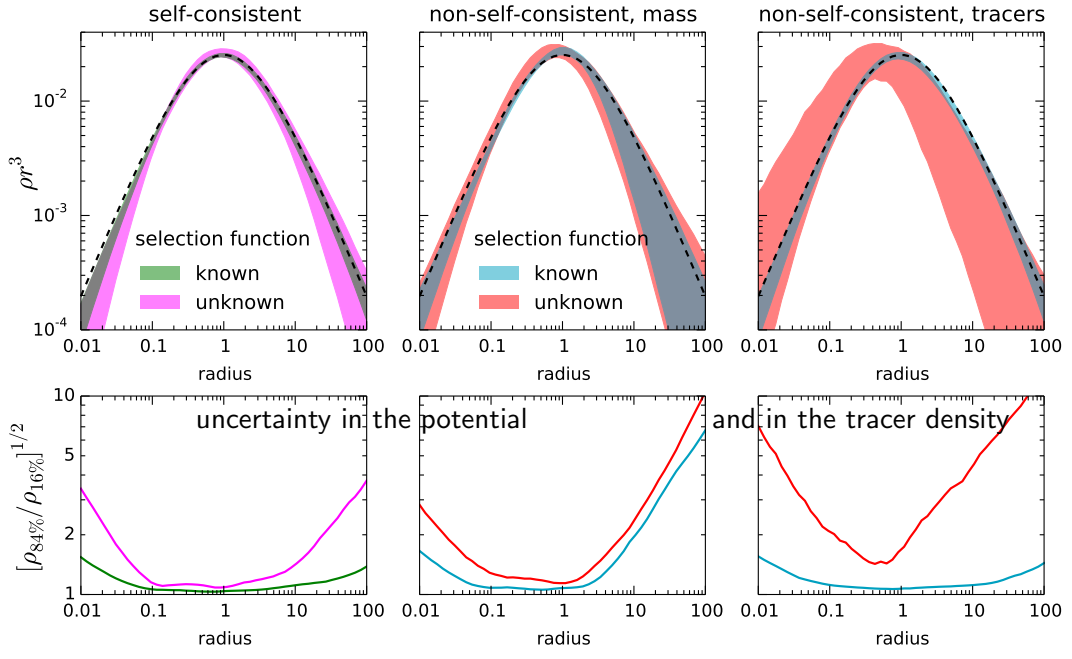
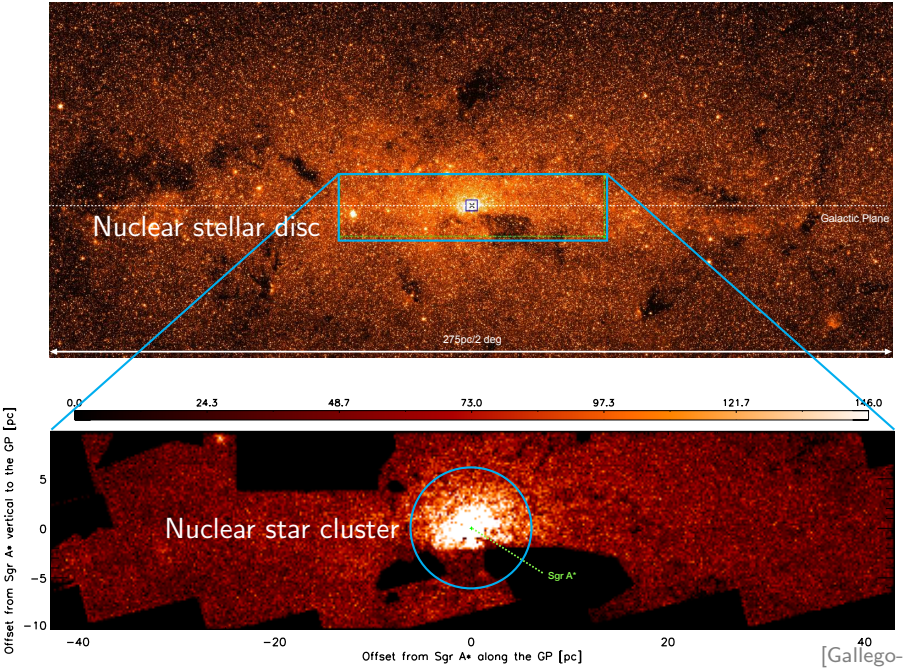
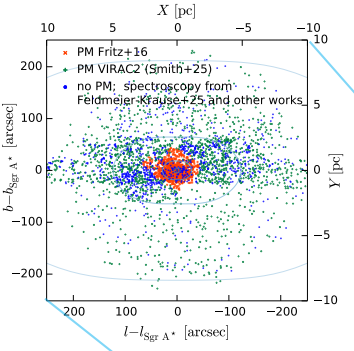


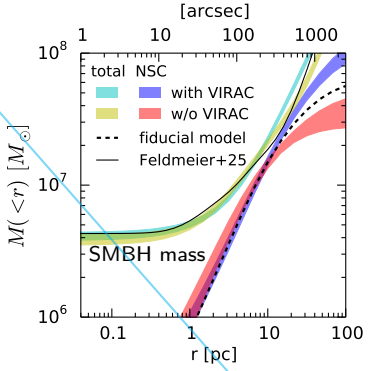
Real application: Milky Way nuclear star cluster

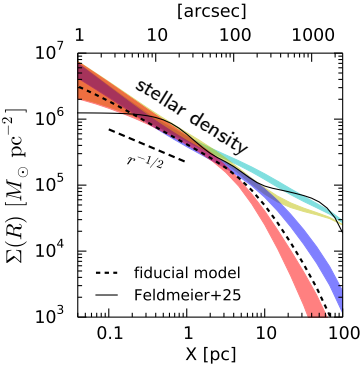


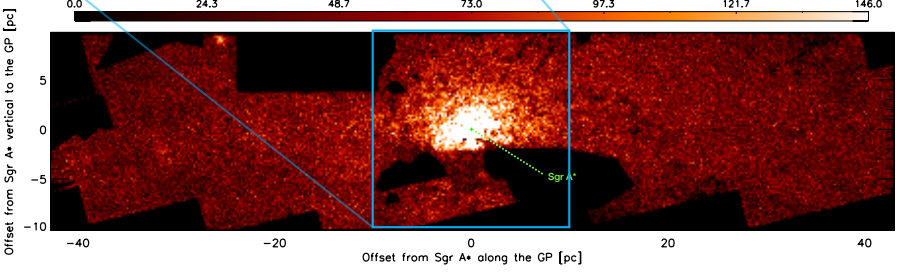
[Gallego-Cano+ 2020]

Real application: Milky Way nuclear star cluster [Vasiliev+ 2026]









Episode 2: measuring the shape of the potential [Gherghinescu+ 2026]

Context:

constrain the shape of the dark matter haloes in external galaxies using discrete-kinematic tracers:

- stars in dSph galaxies around MW, or
- star clusters and PNe in galaxies like M31.

Ingredients:

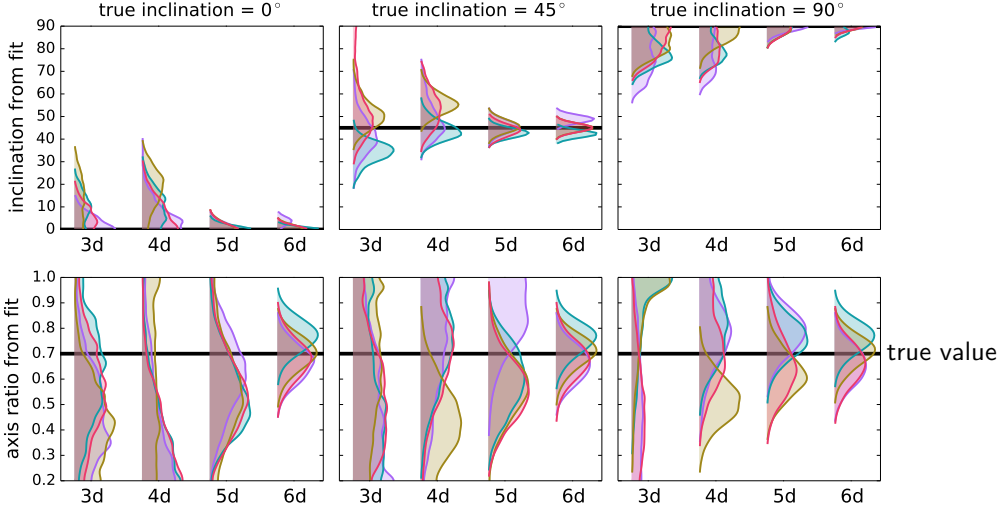
axisymmetric dark halo potential $\Phi_{\rm ext}$ created by an oblate density profile $\rho\big(\sqrt{R^2+(z/q)^2}\big),\ q\leq 1$ (e.g. a generalised NFW) at inclination i;
DF $f(\mathcal{I})$ for the tracer population; not self-consistent.

$N_\star \sim \mathcal{O}(10^3)$ tracers with incomplete phase-space information:
sky coordinates X, Y and

- 3d: $v_{\rm LoS}$ only;
- 4d: sky-plane velocities v_X, v_Y;
- 5d: all three velocity components, but no distance;
- 6d: everything known.

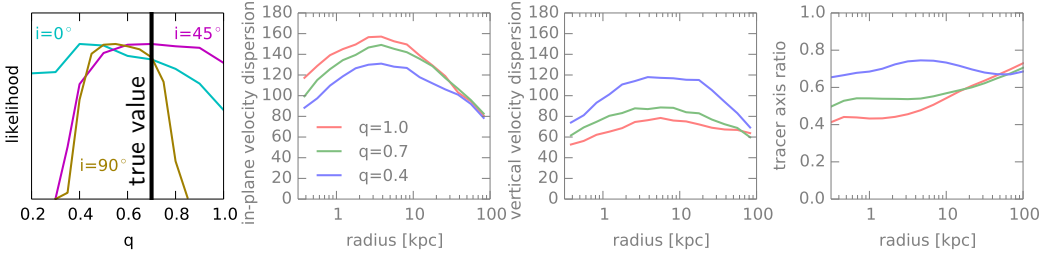
Constraints on the potential from DF fits

- ✓ sphericalised density profile and inclination are well recovered
- × halo shape q is **not** tightly constrained with 3d, 4d or even 5d data
- × large variation between different realisations of mock data



Constraints on the potential from Schwarzschild models

Could it be a limitation of the DF-based method?
another test: Schwarzschild models of integrated-light kinematics
(S/N equivalent to $\mathcal{O}(10^6)$ discrete tracers,
100+ bins with 8 Gauss–Hermite moments,
spatial coverage up to 100 kpc $\gtrsim 10\,R_{\rm eff}$).



Results: no constraints on the axis ratio q for face-on orientation ($i = 0°$), and still quite weak constraints for edge-on ($i = 90°$).
Enough internal freedom in the model to rearrange radial and vertical velocity distributions to keep the same line-of-sight kinematics.

Another test: constraints on shape and inclination

Setup:

Oblate Perfect Ellipsoid potential with axis ratio q, self-consistent (no DM halo); kinematic data (6 GH moments) within region containing 90% of total luminosity; different combinations of q and i giving the same projected axis ratio $\simeq 0.8$; two free parameters – i and mass-to-light ratio M/L.

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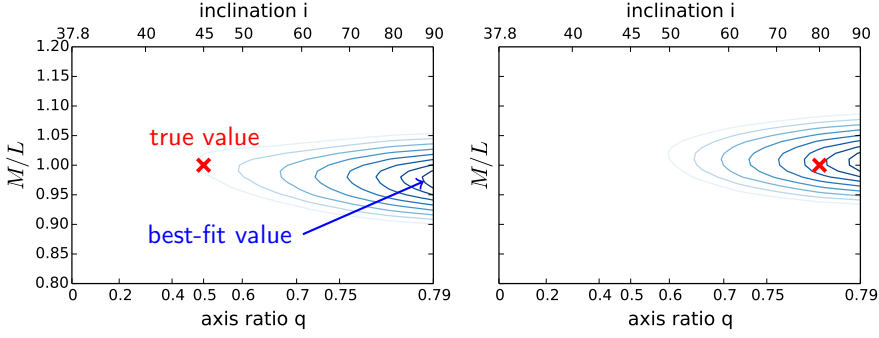
Result:

total disaster – regardless of true i, fits always prefer $i = 90^\circ$ [Thomas+ 2007]



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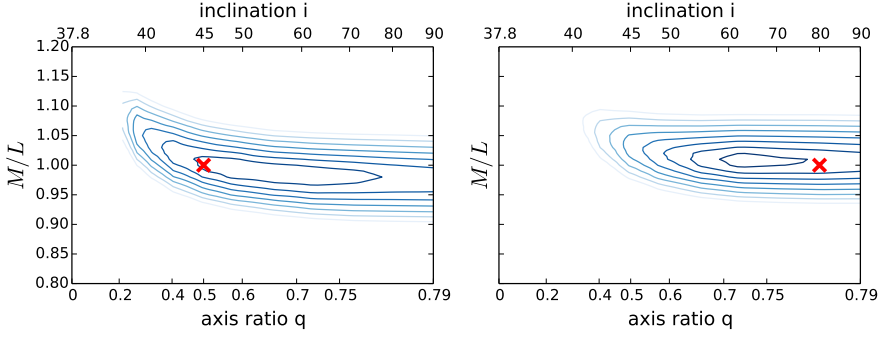
two free parameters – i and mass-to-light ratio M/L.
add the penalty for model flexibility to raw χ^2 values [Lipka & Thomas+ 2021].

Result:

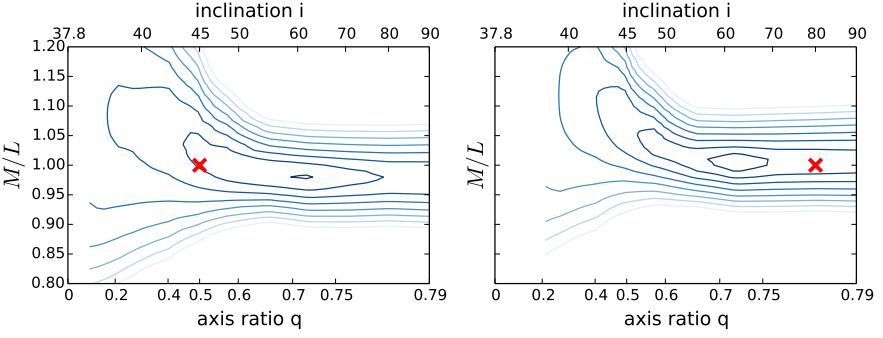
bias is gone, but constraints on i (and hence q) are pretty weak! [see also Krajnović+ 2005]



Another test: constraints on shape and inclination

Setup:

Oblate Perfect Ellipsoid potential with axis ratio $q_{\rm pot}$, **non**-self-consistent:
fit only the surface density profile and kinematics, but not 3d density constraints;
$q_{\rm pot}$ and i are now decoupled, but even if we keep them linked as before,
the constraints are even weaker.



Summary

- One can do meaningful dynamical modelling even without knowing the density profile of kinematic tracers $\Longrightarrow$
  could be useful to sidestep complex selection functions or patchy extinction.
- The shape of an [axisymmetric] potential is difficult to constrain
  (though this does not impact the measurement of a sphericalized mass profile).

  Other studies [de Nicola+ 2022, Neureiter+ 2023; Veršič+ 2024] are more optimistic – need to understand the disagreement...

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[Giuseppe Capogrossi]